

CONSTRUCTION OF IRREDUCIBLE REPRESENTATIONS OVER KHOVANOV-LAUDA-ROUQUIER ALGEBRAS OF FINITE CLASSICAL TYPE

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In Memory of Professor Hyo Chul Myung

ABSTRACT. We give an explicit construction of irreducible modules over Khovanov-Lauda-Rouquier algebras R and their cyclotomic quotients R^λ for finite classical types using a crystal basis theoretic approach. More precisely, for each element v of the crystal $B(\infty)$ (resp. $B(\lambda)$), we first construct certain modules $\nabla(\mathbf{a}; k)$ labeled by the adapted string \mathbf{a} of v . We then prove that the head of the induced module $\text{Ind}(\nabla(\mathbf{a}; 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}; n))$ is irreducible and that every irreducible R -module (resp. R^λ -module) can be realized as the irreducible head of one of the induced modules $\text{Ind}(\nabla(\mathbf{a}; 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}; n))$. Moreover, we show that our construction is compatible with the crystal structure on $B(\infty)$ (resp. $B(\lambda)$).

INTRODUCTION

The *Khovanov-Lauda-Rouquier algebras* (*KLR algebras*) were introduced independently by Khovanov-Lauda [18, 19] and Rouquier [24] to give a categorification of quantum groups. Let $U_q(\mathfrak{g})$ be the quantum group associated with a symmetrizable Cartan datum, and let R be the corresponding KLR algebra. For a dominant integral weight λ of $U_q(\mathfrak{g})$, the algebra R has a special quotient R^λ corresponding to λ , which is called the *cyclotomic Khovanov-Lauda-Rouquier algebra* (*cyclotomic KLR algebra*) of weight λ . It was conjectured that the cyclotomic quotient R^λ gives a categorification of the irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\lambda)$. This was shown for type A_n in [1, 2, 3]. In [11], Kang and Kashiwara proved this conjecture for all symmetrizable Cartan data. Webster [25] has announced a categorification of tensor products of highest weight modules. In a recent paper, Kang, Oh and Park [13] extended the study of KLR algebras to provide a categorification of quantum generalized Kac-Moody algebras and their crystals. Moreover, Kang, Kashiwara and Oh [12] proved the

2000 *Mathematics Subject Classification.* 05E10, 16D60, 17B67, 81R10.

Key words and phrases. adapted strings, crystal bases, Khovanov-Lauda-Rouquier algebras.

¹ This work was supported by KRF Grant # 2007-341-C00001.

² This work was supported by NRF Grant # 2010-0010753.

³ This work was supported by NRF Grant # 2010-0019516.

⁴ This work was supported by BK21 Mathematical Sciences Division.

cyclotomic categorification conjecture for irreducible highest weight modules over quantum generalized Kac-Moody algebras.

For symmetrizable Cartan data, the set $\mathbb{B}(\infty)$ (resp. $\mathbb{B}(\lambda)$) of isomorphism classes of finite-dimensional irreducible graded modules over R (resp. R^λ) can be given a crystal structure, and Lauda and Vazirani [21] have shown that there exist crystal isomorphisms $B(\infty) \xrightarrow{\sim} \mathbb{B}(\infty)$ and $B(\lambda) \xrightarrow{\sim} \mathbb{B}(\lambda)$, where $B(\infty)$ (resp. $B(\lambda)$) is the crystal of $U_q^-(\mathfrak{g})$ (resp. $V(\lambda)$). Kleshchev and Ram [20] gave a construction of irreducible graded R -modules for all finite types by using the combinatorics of Lyndon words to construct irreducible R -modules as the irreducible heads of the induced modules of the outer tensor products of *cuspidal modules*. In this approach, the action of Kashiwara operators on the crystal of irreducible modules is hidden in the combinatorics of Lyndon words. Hill, Melvin and Mondragon [6] completed the classification of irreducible R -modules begun by Kleshchev and Ram by determining the cuspidal modules. However, it remains an open problem to construct irreducible graded R^λ -modules. For type $A_n^{(1)}$, Hu and Mathas [9] constructed a homogeneous cellular basis of R^λ , which yields irreducible modules by using the cellular basis techniques introduced in [5].

In this paper, we give an explicit construction of all irreducible graded modules over R and R^λ for KLR algebras of finite classical type as the irreducible heads of certain induced modules. This generalizes the type A_n result in [14] to all finite classical types. Our construction differs from the one given by Kleshchev and Ram and is based on the theory of crystal bases. As a result, the action of the Kashiwara operators is an integral part of the construction.

Here is a brief description of our work. Let $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ be a symmetrizable Cartan datum. Let I be the index set of the simple roots, and let $n = |I|$. Set $I_{(n+1)} = I$ and take $I_{(k)}$ ($k = 1, \dots, n$) to be subsets of I such that $I_{(k)} \subset I_{(k+1)}$ and $|I_{(k)}| = k$ for all k . Let \mathfrak{B}_k be the crystal obtained from $B(\infty)$ by forgetting the i -arrows for $i \notin I_{(k)}$. For $v \in B(\infty)$, let $u_0 = v$ and let u_k be the highest weight vector of the connected component of \mathfrak{B}_k containing v for $k = 1, \dots, n$. Then, there exists a sequence \mathbf{i}_k of elements in I such that $u_{k-1} = \tilde{f}_{\mathbf{i}_k} u_k$, where $\tilde{f}_{\mathbf{i}_k}$ is a product of the Kashiwara operators corresponding to the terms in the sequence \mathbf{i}_k . For $k = 1, \dots, n$, define

$$\mathcal{N}_k(v) = \tilde{f}_{\mathbf{i}_k} \mathbf{1},$$

where $\mathbf{1}$ is the trivial for the KLR algebra $R(0) := \mathbb{C}$. In Proposition 1.10, we prove for any symmetrizable Cartan datum that $\text{hd lnd}(\mathcal{N}_1(v) \boxtimes \cdots \boxtimes \mathcal{N}_n(v))$ is an irreducible graded R -module and that the map $\Phi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ defined by

$$(0.1) \quad \Phi(v) = \text{hd lnd}(\mathcal{N}_1(v) \boxtimes \cdots \boxtimes \mathcal{N}_n(v)) \quad \text{for } v \in B(\infty)$$

is a crystal isomorphism.

When \mathfrak{A} is of finite type, the modules $\mathcal{N}_k(v)$ can be given a more explicit description in terms of the Kashiwara operators \tilde{f}_i via the *adapted strings* introduced by Littelmann in [23]. For this, we choose a special expression $w_0 = r_{\mathbf{s}_1} \cdots r_{\mathbf{s}_n}$ of the longest element w_0 in the Weyl group W . Here the \mathbf{s}_k are sequences of indices in I (see Table 1). Using the description of adapted strings given in [23], we prove for $v \in B(\infty)$ that

$$\mathcal{N}_k(v) = \tilde{f}_{\mathbf{s}_k}^{\mathbf{a}(v)_k} \mathbf{1},$$

where $\mathbf{a}(v)$ is the adapted string of v with respect to this expression for w_0 , and $\mathbf{a}(v)_k$ is the subsequence of $\mathbf{a}(v)$ defined by (2.3) (see Proposition 2.3 below).

Our main result is an explicit construction of the irreducible graded modules over the KLR algebras R and R^λ of finite classical type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$ in terms of adapted strings. Let \mathcal{S} (resp. \mathcal{S}^λ) be the set of adapted strings of $B(\infty)$ (resp. $B(\lambda)$) given in Proposition 3.1, and let \mathbf{B} be the crystal of the irreducible module $V(\Lambda_n)$ labeled by the fundamental weight Λ_n if \mathfrak{A} is of type \mathbf{A}_n (resp. $V(\Lambda_1)$ if \mathfrak{A} is of type $\mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$). For $a, b \in \mathbf{B}$ with $a \succ b$, using the structure of the crystal \mathbf{B} , in (3.3) we associate to a, b an irreducible graded module $\nabla_{(a,b)}$. The modules $\nabla_{(a,b)}$ are 1- or 2-dimensional, and in general they are *not* the cuspidal modules found in [6, 20]. Using the description of \mathcal{S} (resp. \mathcal{S}^λ), we define for $v \in B(\infty)$ (resp. $v \in B(\lambda)$) the module $\nabla(\mathbf{a}(v); k)$ to be the outer tensor product of modules $\nabla_{(a,b)}$ as in (3.5). Then, it follows from Lemma 4.3 that

$$(0.2) \quad \mathcal{N}_k(v) = \text{hd lnd} \nabla(\mathbf{a}(v); k) \quad \text{for } k = 1, \dots, n.$$

The description of the irreducible modules in (0.1) works for all symmetrizable Cartan data. It follows from (0.1) and (0.2) that the irreducible modules for the KLR algebras of finite classical type can be obtained from taking the outer tensor product of heads of induced modules, inducing, then taking the head of the induced module. But this unduly complicated process can be simplified. Indeed, we prove for the module $\nabla(\mathbf{a}(v)) := \nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)$ that $\text{hd lnd} \nabla(\mathbf{a}(v))$ is irreducible for $v \in B(\infty)$ (resp. $v \in B(\lambda)$) for all finite classical types and that the maps

$$\begin{aligned} \Psi : B(\infty) &\longrightarrow \mathbb{B}(\infty) & \text{given by } \Psi(v) &= \text{hd lnd} \nabla(\mathbf{a}(v)) \text{ for } v \in B(\infty), \\ \Psi^\lambda : B(\lambda) &\longrightarrow \mathbb{B}(\lambda) & \text{given by } \Psi^\lambda(v) &= \text{hd lnd} \nabla(\mathbf{a}(v)) \text{ for } v \in B(\lambda) \end{aligned}$$

are crystal isomorphisms (Theorem 3.2). Using that fact, we show for the finite classical types that

$$\begin{aligned} \mathcal{A} &= \{\text{hd lnd} \nabla(\mathbf{a}) \mid \mathbf{a} \in \mathcal{S}\} \text{ and} \\ \mathcal{A}^\lambda &= \{\text{hd lnd} \nabla(\mathbf{a}) \mid \mathbf{a} \in \mathcal{S}^\lambda\}, \end{aligned}$$

as \mathbf{a} ranges over the adapted strings, provide complete lists of all the irreducible graded modules over R and R^λ , respectively, up to isomorphism and grading shift (Corollary 3.3).

Our paper is organized as follows. Section 1 contains a brief review of crystal bases and KLR algebras associated with any symmetrizable Cartan datum. This section culminates with the proof of the crystal isomorphism $\Phi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ in (0.1). Section 2 specializes to the case of finite type, first reviewing Littelmann's result (see [23]) on adapted strings and then giving an explicit description of the modules $\mathcal{N}_k(v)$ for all finite types via adapted strings. Combining this description with the definition of $\mathcal{N}_k(v)$, we obtain an expression for $\mathcal{N}_k(v)$ in terms of Kashiwara operators.

In Section 3, we restrict to the case of finite classical types. We first define the modules $\nabla_{(a,b)}$ using the structure of the crystals \mathbf{B} and construct the module $\nabla(\mathbf{a}; k)$ for $\mathbf{a} \in \mathcal{S}$ (resp. $\mathbf{a} \in \mathcal{S}^\lambda$) as an outer tensor product of modules $\nabla_{(a,b)}$ in (3.5). We then construct the maps $\Psi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ and $\Psi^\lambda : B(\lambda) \longrightarrow \mathbb{B}(\lambda)$ by taking the head of $\text{Ind} \nabla(\mathbf{a}(v))$ for $v \in B(\infty)$ (resp. $v \in B(\lambda)$). We illustrate this construction by presenting an example for type B_3 using the Kashiwara-Nakashima tableaux in [17] to realize the crystal $B(\lambda)$. We prove in Proposition 3.4 for any finite classical type and any $v \in B(\lambda)$ that the number $\eta(v)$ of $\nabla_{(a,b)}$'s in $\nabla(\mathbf{a}(v))$ has an upper bound; i.e., $\eta(v) \leq n\lambda(h)$ for a certain element $h \in \mathbf{P}^\vee$ (which depends on the type).

Section 4 is devoted to proving that the maps $\Psi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ and $\Psi^\lambda : B(\lambda) \longrightarrow \mathbb{B}(\lambda)$ are crystal isomorphisms. To accomplish this, we give a sufficient condition in Lemma 4.1 for the isomorphism $\text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(c,d)}) \simeq \text{Ind}(\nabla_{(c,d)} \boxtimes \nabla_{(a,b)})$ to exist for finite classical types. Using this condition together with Lemma 4.3 of [14] and the surjective homomorphism $\text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(b,c)}) \twoheadrightarrow \nabla_{(a,c)}$, we prove in Lemma 4.3 that $\mathcal{N}_n(v) = \text{hd} \text{Ind} \nabla(\mathbf{a}(v); n)$. It follows from that result and the choice of the sequence corresponding to k that $\mathcal{N}_k(v) = \text{hd} \text{Ind} \nabla(\mathbf{a}(v); k)$ for all $k = 1, \dots, n$. Combining this with the crystal isomorphism Φ in (0.1), we establish the crystal isomorphisms $\Psi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ and $\Psi^\lambda : B(\lambda) \longrightarrow \mathbb{B}(\lambda)$. This then gives an explicit realization compatible with the Kashiwara operators of all the irreducible graded modules over R and R^λ for finite classical types.

1. CRYSTALS AND KHOVANOV-LAUDA-ROUQUIER ALGEBRAS

1.1. Crystals.

Let I be a finite index set. A square matrix $\mathfrak{A} = (a_{ij})_{i,j \in I}$ is a *symmetrizable generalized Cartan matrix* if it satisfies (i) $a_{ii} = 2$ for $i \in I$, (ii) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$, (iii) $a_{ij} = 0$ if $a_{ji} = 0$ for $i, j \in I$, (iv) there is a diagonal matrix $\mathfrak{D} = \text{diag}(\delta_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $\mathfrak{D}\mathfrak{A}$ is symmetric.

A *Cartan datum* $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ consists of

- (1) a symmetrizable generalized Cartan matrix \mathfrak{A} ,

- (2) a free abelian group P of finite rank, called the *weight lattice*,
- (3) the set $\Pi = \{\alpha_i \mid i \in I\} \subset P$ of *simple roots*,
- (4) the *dual weight lattice* $P^\vee := \text{Hom}(P, \mathbb{Z})$,
- (5) the set $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$ of *simple coroots*,

which satisfy the following properties:

- (i) $\langle h_i, \alpha_j \rangle := \alpha_j(h_i) = a_{ij}$ for all $i, j \in I$,
- (ii) $\Pi \subset \mathfrak{h}^*$ is linearly independent, where $\mathfrak{h} := \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ and \mathfrak{h}^* is the dual space,
- (iii) for each $i \in I$, there exists $\Lambda_i \in P$ such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$ for all $j \in I$.

The Λ_i are the *fundamental weights*. We denote by $P^+ = \{\lambda \in P \mid \lambda(h_i) \in \mathbb{Z}_{\geq 0}, i \in I\}$ the set of *dominant integral weights*. The free abelian group $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ is the *root lattice*, and $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the *positive root lattice*. For $\alpha = \sum_{i \in I} k_i \alpha_i \in Q^+$, the *height* of α is $|\alpha| := \sum_{i \in I} k_i$. There is a symmetric bilinear form (\mid) on \mathfrak{h}^* such that

$$(\alpha_i \mid \alpha_j) = \delta_{ij} a_{ij} \text{ for } i, j \in I, \quad \langle h_i, \lambda \rangle = \frac{2(\alpha_i \mid \lambda)}{(\alpha_i \mid \alpha_i)} \text{ for } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$

Let W be the *Weyl group*, which is the subgroup of $\text{Aut}(\mathfrak{h}^*)$ generated by simple reflections $\{r_i\}_{i \in I}$ defined by $r_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i$ for $\lambda \in \mathfrak{h}^*$ and $i \in I$.

Let q be an indeterminate. For $i \in I$ and $m, n \in \mathbb{Z}_{\geq 0}$, define

$$q_i = q^{\delta_i}, \quad [n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}, \quad [n]_{q_i}! = \prod_{k=1}^n [k]_{q_i}, \quad \begin{bmatrix} m \\ n \end{bmatrix}_{q_i} = \frac{[m]_{q_i}!}{[m-n]_{q_i}! [n]_{q_i}!}.$$

Definition 1.1. The *quantum group* $U_q(\mathfrak{g})$ associated with the Cartan datum $(\mathfrak{A}, P, \Pi, P^\vee, \Pi^\vee)$ is the associative algebra over $\mathbb{Q}(q)$ with 1 generated by e_i, f_i ($i \in I$) and q^h ($h \in P^\vee$) satisfying the following relations:

- (1) $q^0 = 1$, $q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$,
- (2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$, $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $h \in P^\vee, i \in I$,
- (3) $e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$, where $K_i = q^{\delta_i h_i}$,
- (4) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \neq j$,
- (5) $\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0$ if $i \neq j$.

The definition of the *category* O_{int}^q of integrable $U_q(\mathfrak{g})$ -modules, *crystal bases*, and *Kashiwara operators* can be found, for example, in [7, 15]. It was proved in [15] that every $U_q(\mathfrak{g})$ -module in the category O_{int}^q has a unique crystal basis (L, B) . Let us recall the notion of a crystal as defined in [16].

Definition 1.2. A *crystal* is a set B together with maps $\text{wt} : B \rightarrow \mathbf{P}$, $\varphi_i, \varepsilon_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \rightarrow B \sqcup \{0\}$ ($i \in I$) which satisfy the following conditions:

- (1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle$,
- (2) $\text{wt}(\tilde{e}_i b) = \text{wt}(b) + \alpha_i$, $\text{wt}(\tilde{f}_i b) = \text{wt}(b) - \alpha_i$ if $\tilde{e}_i b, \tilde{f}_i b \in B$,
- (3) for $b, b' \in B$ and $i \in I$, $b' = \tilde{e}_i b$ if and only if $b = \tilde{f}_i b'$,
- (4) for $b \in B$, if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$,
- (5) if $b \in B$ and $\tilde{e}_i b \in B$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
- (6) if $b \in B$ and $\tilde{f}_i b \in B$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$.

For $\mathbf{i} = (i_1, \dots, i_m) \in I^m$ and $\mathbf{k} = (k_1, \dots, k_m) \in (\mathbb{Z}_{\geq 0})^m$, let $\tilde{f}_{\mathbf{i}}^{\mathbf{k}} = \tilde{f}_{i_1}^{k_1} \cdots \tilde{f}_{i_m}^{k_m}$ (resp. $\tilde{e}_{\mathbf{i}}^{\mathbf{k}} = \tilde{e}_{i_1}^{k_1} \cdots \tilde{e}_{i_m}^{k_m}$). If $\mathbf{k} = (1, \dots, 1)$, then we write $\tilde{f}_{\mathbf{i}}$ (resp. $\tilde{e}_{\mathbf{i}}$) for $\tilde{f}_{\mathbf{i}}^{\mathbf{k}}$ (resp. $\tilde{e}_{\mathbf{i}}^{\mathbf{k}}$).

Examples 1.3.

- (1) Associated with each module $M \in O_{int}^q$ is a crystal basis (L, B) , where B is a crystal with the maps

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \max\{k \geq 0 \mid \tilde{f}_i^k b \neq 0\}.$$

We denote by $B(\lambda)$ the crystal of the irreducible highest weight module $V(\lambda) \in O_{int}^q$ with highest weight $\lambda \in \mathbf{P}^+$ and write b_λ for the highest weight element of $B(\lambda)$.

- (2) Let $(L(\infty), B(\infty))$ be the crystal basis of $U_q^-(\mathfrak{g})$. Then $B(\infty)$ is a crystal with the maps given by

$$\varepsilon_i(b) = \max\{k \geq 0 \mid \tilde{e}_i^k b \neq 0\}, \quad \varphi_i(b) = \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle.$$

Let 1 denote the highest weight element of $B(\infty)$.

- (3) For $\lambda \in \mathbf{P}$, the set $T_\lambda = \{t_\lambda\}$ is a crystal with

$$\begin{aligned} \text{wt}(t_\lambda) &= \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for } i \in I, \\ \varepsilon_i(t_\lambda) &= \varphi_i(t_\lambda) = -\infty \quad \text{for } i \in I. \end{aligned}$$

- (4) The set $C = \{c\}$ is a crystal with

$$\text{wt}(c) = 0, \quad \tilde{e}_i c = \tilde{f}_i c = 0, \quad \varepsilon_i(c) = \varphi_i(c) = 0 \quad (i \in I).$$

We refer to [7, 16] for more details on $U_q(\mathfrak{g})$ -crystals. It was shown in [10, 16] that for each $\lambda \in \mathbf{P}^+$ there is a unique strict crystal embedding

$$(1.1) \quad \iota_\lambda : B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \otimes C$$

sending b_λ to $1 \otimes t_\lambda \otimes c$.

1.2. Khovanov-Lauda-Rouquier algebras.

For $\alpha \in \mathbb{Q}^+$ with $|\alpha| = m$, we define $I^\alpha = \{\mathbf{i} = (i_1, \dots, i_m) \in I^m \mid \sum_{k=1}^m \alpha_{i_k} = \alpha\}$. Given $\mathbf{i} = (i_1, \dots, i_m) \in I^\alpha$ and $\mathbf{j} = (j_1, \dots, j_{m'}) \in I^\beta$, let $\mathbf{i} * \mathbf{j}$ denote the concatenation of \mathbf{i} and \mathbf{j} : $\mathbf{i} * \mathbf{j} = (i_1, \dots, i_m, j_1, \dots, j_{m'}) \in I^{\alpha+\beta}$. Let S_m be the symmetric group on m letters with simple transpositions σ_i ($i = 1, \dots, m-1$). Then S_m acts on I^m in a natural way. For $d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}$, we denote by $S_{d_1+\dots+d_n}/S_{d_1} \times \dots \times S_{d_n}$ the set of minimal left coset representatives of $S_{d_1} \times \dots \times S_{d_n}$ in $S_{d_1+\dots+d_n}$.

Let u, v be indeterminates. For each $i, j \in I$, we choose $\zeta_{ij} \in \mathbb{C} \setminus \{0\}$ such that $\zeta_{ij} = \zeta_{ji}$ if $a_{ij} = 0$ and elements $\eta_{ij} \in \mathbb{C}$ with $\eta_{ij}^{pq} = \eta_{ji}^{qp}$ for all $p, q \in \mathbb{Z}_{>0}$ such that $\delta_i p + \delta_j q = -(\alpha_i | \alpha_j)$ and set

$$\mathcal{Q}_{ij}(u, v) = \begin{cases} 0 & \text{if } i = j, \\ \zeta_{ij} & \text{if } i \neq j, a_{ij} = 0, \\ \zeta_{ij} u^{-a_{ij}} + \sum_{\substack{p, q > 0, \\ \delta_i p + \delta_j q = -(\alpha_i | \alpha_j)}} \eta_{ij}^{pq} u^p v^q + \zeta_{ji} v^{-a_{ji}} & \text{otherwise.} \end{cases}$$

Definition 1.4.

- (1) Let $\alpha \in \mathbb{Q}^+$ with $|\alpha| = m$. The *homogeneous Khovanov-Lauda-Rouquier algebra* $R(\alpha)$ at α associated with \mathfrak{A} and $(\mathcal{Q}_{ij})_{i, j \in I}$ is the associative graded \mathbb{C} -algebra generated by $e(\mathbf{i})$ ($\mathbf{i} = (i_1, \dots, i_m) \in I^\alpha$), x_ℓ ($1 \leq \ell \leq m$), τ_k ($1 \leq k < m$) satisfying the following defining relations:

$$\begin{aligned} e(\mathbf{i})e(\mathbf{j}) &= \delta_{\mathbf{i}, \mathbf{j}}e(\mathbf{i}), \quad \sum_{\mathbf{i} \in I^\alpha} e(\mathbf{i}) = 1, \quad x_k x_\ell = x_\ell x_k, \quad x_\ell e(\mathbf{i}) = e(\mathbf{i})x_\ell, \\ \tau_k e(\mathbf{i}) &= e(\sigma_k(\mathbf{i}))\tau_k, \quad \tau_k \tau_\ell = \tau_\ell \tau_k \quad \text{if } |k - \ell| > 1, \\ \tau_k^2 e(\mathbf{i}) &= \mathcal{Q}_{i_k, i_{k+1}}(x_k, x_{k+1})e(\mathbf{i}), \\ (\tau_k x_\ell - x_{\sigma_k(\ell)} \tau_k) e(\mathbf{i}) &= \begin{cases} -e(\mathbf{i}) & \text{if } \ell = k, \quad i_k = i_{k+1}, \\ e(\mathbf{i}) & \text{if } \ell = k+1, \quad i_k = i_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \\ (\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\mathbf{i}) &= \begin{cases} \frac{\mathcal{Q}_{i_k, i_{k+1}}(x_{k+2}, x_{k+1}) - \mathcal{Q}_{i_k, i_{k+1}}(x_k, x_{k+1})}{x_{k+2} - x_k} e(\mathbf{i}) & \text{if } i_k = i_{k+2} \neq i_{k+1}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The algebra

$$R := \bigoplus_{\alpha \in \mathbb{Q}^+} R(\alpha)$$

is called the *Khovanov-Lauda-Rouquier algebra* (*KLR algebra*) associated with \mathfrak{A} and $(\mathcal{Q}_{ij})_{i,j \in I}$.

- (2) Let $\lambda \in P^+$. The *homogeneous cyclotomic Khovanov-Lauda-Rouquier algebra* $R^\lambda(\alpha)$ at α of weight λ is the quotient algebra of $R(\alpha)$ by the two-sided ideal $I^\lambda(\alpha)$ of $R(\alpha)$ generated by $x_m^{(h_{im}, \lambda)} e(\mathbf{i})$ ($\mathbf{i} \in I^\alpha$). The algebra

$$R^\lambda := \bigoplus_{\alpha \in Q^+} R^\lambda(\alpha)$$

is the *cyclotomic Khovanov-Lauda-Rouquier algebra* (*cyclotomic KLR algebra*) of weight λ .

The \mathbb{Z} -grading on $R(\alpha)$ is given by

$$\deg(e(\mathbf{i})) = 0, \quad \deg(x_\ell e(\mathbf{i})) = (\alpha_{i_\ell} | \alpha_{i_\ell}), \quad \deg(\tau_k e(\mathbf{i})) = -(\alpha_{i_k} | \alpha_{i_{k+1}}).$$

Here $R(0) = \mathbb{C}$. Let $R(\alpha)\text{-fmod}$ (resp. $R^\lambda(\alpha)\text{-fmod}$) be the category of finite-dimensional \mathbb{Z} -graded $R(\alpha)$ -modules (resp. $R^\lambda(\alpha)$ -modules). Any $N \in R^\lambda(\alpha)\text{-fmod}$ can be viewed as a graded $R(\alpha)$ -module annihilated by $I^\lambda(\alpha)$. We write $\text{infl}^\lambda N$ when considering $N \in R^\lambda(\alpha)\text{-fmod}$ as an $R(\alpha)$ -module. Any $M \in R(\alpha)\text{-fmod}$ gives rise to the $R^\lambda(\alpha)$ -module $\text{pr}^\lambda M := M/I^\lambda(\alpha)M$. From now on, when there is no possibility of confusion, we identify irreducible graded $R^\lambda(\alpha)$ -modules with irreducible graded $R(\alpha)$ -modules annihilated by $I^\lambda(\alpha)$ via infl^λ .

Let

$$G_0(R) = \bigoplus_{\alpha \in Q^+} G_0(R(\alpha)\text{-fmod}), \quad G_0(R^\lambda) = \bigoplus_{\alpha \in Q^+} G_0(R^\lambda(\alpha)\text{-fmod}),$$

where $G_0(R(\alpha)\text{-fmod})$ (resp. $G_0(R^\lambda(\alpha)\text{-fmod})$) is the Grothendieck group of $R(\alpha)\text{-fmod}$ (resp. of $R^\lambda(\alpha)\text{-fmod}$). For $M \in R(\alpha)\text{-fmod}$ (resp. $R^\lambda(\alpha)\text{-fmod}$), $[M]$ stands for the isomorphism class of M in $G_0(R(\alpha))$ (resp. $G_0(R^\lambda(\alpha))$). When no confusion can arise, we write M for $[M]$.

Given $M = \bigoplus_{i \in \mathbb{Z}} M_i$, let $M\langle k \rangle = \bigoplus_{i \in \mathbb{Z}} M\langle k \rangle_i$ denote the graded module obtained from M by shifting the grading by k , where $M\langle k \rangle_i := M_{i-k}$ for $i \in \mathbb{Z}$. The q -character $\text{ch}_q(M)$ and character $\text{ch}(M)$ of M are defined by

$$\text{ch}_q(M) := \sum_{\mathbf{i} \in I^\alpha} \dim_q(e(\mathbf{i})M) \mathbf{i}, \quad \text{ch}(M) := \sum_{\mathbf{i} \in I^\alpha} \dim(e(\mathbf{i})M) \mathbf{i},$$

where $\dim_q(N) := \sum_{i \in \mathbb{Z}} (\dim N_i) q^i$ for any graded module $N = \bigoplus_{i \in \mathbb{Z}} N_i$. For $\mathbf{i} \in I^\alpha$, we write $\mathbf{i} \in \text{ch}(M)$ (resp. $\mathbf{i} \in \text{ch}_q(M)$) when \mathbf{i} occurs in $\text{ch}M$ (resp. $\text{ch}_q(M)$) with a nonzero coefficient.

Notation and Conventions. In this paper, by a homomorphism we mean a homogeneous homomorphism of some degree $k \in \mathbb{Z}$. The symbol \simeq (resp. \twoheadrightarrow , \hookrightarrow) will be used to denote

an isomorphism (resp. a surjective homomorphism, an injective homomorphism) up to a grading shift, and the notation \cong will be reserved for a degree preserving isomorphism. For $M \in R(\alpha)\text{-fmod}$ and $N \in R(\beta)\text{-fmod}$, $M \boxtimes N$ will denote the outer tensor product of M and N .

For $M, N \in R(\alpha)\text{-fmod}$, let $\text{Hom}(M, N)$ be the \mathbb{C} -vector space of homogeneous homomorphisms of degree 0, and let $\text{HOM}(M, N) = \bigoplus_{k \in \mathbb{Z}} \text{Hom}(M, N\langle k \rangle)$. For $\beta_1, \dots, \beta_m \in \mathbb{Q}^+$, we define

$$(1.2) \quad e(\beta_1, \dots, \beta_m) = \sum_{\mathbf{i}_j \in I^{\beta_j}} e(\mathbf{i}_1 * \dots * \mathbf{i}_m).$$

The natural embedding $R(\beta_1) \otimes \dots \otimes R(\beta_m) \hookrightarrow R(\beta_1 + \dots + \beta_m)$ gives the following functors:

$$\begin{aligned} \text{Ind}_{\beta_1, \dots, \beta_m} _ &:= R(\beta_1 + \dots + \beta_m) \otimes_{R(\beta_1) \otimes \dots \otimes R(\beta_m)} _, \\ \text{Coind}_{\beta_1, \dots, \beta_m} _ &:= \text{HOM}_{R(\beta_1) \otimes \dots \otimes R(\beta_m)}(R(\beta_1 + \dots + \beta_m), _), \\ \text{Res}_{\beta_1, \dots, \beta_m} _ &:= e(\beta_1, \dots, \beta_m) _. \end{aligned}$$

Lemma 1.5 ([21]). (1) Let $M_k \in R(\beta_k)\text{-fmod}$ for $k = 1, \dots, m$. Then

$$\text{Ind}_{\beta_1, \dots, \beta_m}(M_1 \boxtimes \dots \boxtimes M_m) \simeq \text{Coind}_{\beta_m, \dots, \beta_1}(M_m \boxtimes \dots \boxtimes M_1).$$

(2) For $M \in R(\beta_1) \otimes \dots \otimes R(\beta_m)\text{-fmod}$ and $N \in R(\beta_1 + \dots + \beta_m)\text{-fmod}$,

$$\begin{aligned} \text{HOM}_{R(\beta_1 + \dots + \beta_m)}(\text{Ind}_{\beta_1, \dots, \beta_m} M, N) &\cong \text{HOM}_{R(\beta_1) \otimes \dots \otimes R(\beta_m)}(M, \text{Res}_{\beta_1, \dots, \beta_m} N), \\ \text{HOM}_{R(\beta_1 + \dots + \beta_m)}(N, \text{Coind}_{\beta_1, \dots, \beta_m} M) &\cong \text{HOM}_{R(\beta_1) \otimes \dots \otimes R(\beta_m)}(\text{Res}_{\beta_1, \dots, \beta_m} N, M). \end{aligned}$$

For $m \in \mathbb{Z}_{\geq 0}$ and $i \in I$, let

$$L(i^m) = \text{Ind}_{\mathbb{C}[x_1, \dots, x_m]}^{R(m\alpha_i)} \phi,$$

where ϕ is the 1-dimensional trivial module over $\mathbb{C}[x_1, \dots, x_m]$ with $x_j \phi = 0$ for all $j = 1, \dots, m$, and $\dim_q(\phi) = 1$. Let $\mathbb{B}(\infty)$ (resp. $\mathbb{B}(\lambda)$) be the set of all isomorphism classes of irreducible graded R -modules (resp. R^λ -modules). Let $\mathbf{1}$ denote the 1-dimensional trivial $R(0)$ -module. For a graded $R(\beta)$ -module (resp. $R^\lambda(\beta)$ -module) M , it was shown in [11, 18, 21] that the operators defined by

$$\begin{aligned} e_i(M) &= \text{Res}_{\beta - \alpha_i}^{\alpha_i, \beta - \alpha_i} (e(\alpha_i, \beta - \alpha_i)M), \\ f_i(M) &= \begin{cases} R(\beta + \alpha_i)e(\alpha_i, \beta) \otimes_{R(\beta)} M & \text{if } M \in R(\beta)\text{-mod}, \\ R^\lambda(\beta + \alpha_i)e(\alpha_i, \beta) \otimes_{R^\lambda(\beta)} M & \text{if } M \in R^\lambda(\beta)\text{-mod}. \end{cases} \end{aligned}$$

satisfy

$$(1.3) \quad \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} [M] = 0, \quad \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} [M] = 0,$$

where $e_i^{(r)}[M] := \frac{1}{[r]_{q_i}!} [e_i^r M]$ and $f_i^{(r)}[M] := \frac{1}{[r]_{q_i}!} [f_i^r M]$ for $i, j \in I$, $r \in \mathbb{Z}_{\geq 0}$.

We define

$$\begin{aligned} \text{wt}(M) &= \begin{cases} -\beta & \text{if } M \in R(\beta)\text{-fmod}, \\ \lambda - \beta & \text{if } M \in R^\lambda(\beta)\text{-fmod}, \end{cases} \\ \varepsilon_i(M) &= \max\{k \geq 0 \mid e_i^k M \neq 0\}, \quad \varphi_i(M) = \varepsilon_i(M) + \text{wt}(M)(h_i), \\ \tilde{e}_i(M) &= \text{soc}(e_i M), \quad \text{and} \quad \tilde{f}_i(M) = \begin{cases} \text{hd Ind}_{\alpha_i, \beta}(L(i) \boxtimes M) & \text{if } M \in R(\beta)\text{-fmod}, \\ \text{pr}^\lambda \circ \tilde{f}_i \circ \text{infl}^\lambda M & \text{if } M \in R^\lambda(\beta)\text{-fmod}. \end{cases} \end{aligned}$$

The Kashiwara operators \tilde{e}_i and \tilde{f}_i are defined in the opposite manner in [14, 18]. We have made this change so they can be viewed as operators acting on the left of irreducible modules. In Section 3, we will explain further why we want the Kashiwara operators defined in this way.

Theorem 1.6 ([21, Thm. 7.4, Thm. 7.5]). *The sextuple $(\mathbb{B}(\infty), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$ (resp. $(\mathbb{B}(\lambda), \text{wt}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i)$) is a crystal, which is isomorphic to the crystal $B(\infty)$ (resp. $B(\lambda)$).*

1.3. Construction of irreducible R -modules.

In this subsection, we give a construction of irreducible graded R -modules as the irreducible heads of certain induced modules, which is compatible with the Kashiwara operators. More precisely, set $I_{(n+1)} = I$ and let $I_{(k)} \subset I$ ($k = 1, \dots, n$) be subsets of I such that $I_{(k)} \subseteq I_{(k+1)}$ and $|I_{(k)}| = k$. Let \mathfrak{B}_k be the crystal obtained from $B(\infty)$ by forgetting the i -arrows for $i \notin I_{(k)}$. Using highest weight vectors of connected components of \mathfrak{B}_k , we define irreducible R -modules $\mathcal{N}_k(v)$ for $k = 1, \dots, n$ and $v \in B(\infty)$. Then, we construct a crystal isomorphism $\Phi : B(\infty) \rightarrow \mathbb{B}(\infty)$ defined by $\Phi(v) = \text{hd Ind}(\mathcal{N}_1(v) \boxtimes \dots \boxtimes \mathcal{N}_n(v))$ (Proposition 1.10). We emphasize that this crystal isomorphism Φ exists for an arbitrary symmetrizable Cartan datum.

For $M \in R(\beta)\text{-fmod}$, $M_k \in R(\beta_k)\text{-fmod}$ ($k = 1, \dots, m$) and $d \in \mathbb{Z}_{>0}$, set

$$M^{\boxtimes 0} = \mathbb{C}, \quad M^{\boxtimes d} = \underbrace{M \boxtimes \dots \boxtimes M}_d \quad \boxtimes_{k=1}^m M_k = M_1 \boxtimes \dots \boxtimes M_m.$$

Lemma 1.7. *Let $M_k \in R(\beta_k)\text{-fmod}$, $d_k \in \mathbb{Z}_{\geq 0}$, and $\mathbf{i}_k \in I^{d_k \beta_k}$ for $k = 1, \dots, m$. Suppose that \mathbf{i}_k occurs in $\text{ch Ind}(M_k^{\boxtimes d_k})$ with multiplicity $\xi_k \in \mathbb{Z}_{>0}$ for $k = 1, \dots, m$. Assume*

- (i) $\text{Ind}(M_k \boxtimes M_{k'}) \simeq \text{Ind}(M_{k'} \boxtimes M_k)$ for $k, k' = 1, \dots, m$,
- (ii) $\text{Ind}(M_k^{\boxtimes d_k})$ is irreducible for $k = 1, \dots, m$, and
- (iii) $\mathbf{i} := \mathbf{i}_1 * \dots * \mathbf{i}_m$ occurs in $\text{ch Ind}\left(\boxtimes_{k=1}^m M_k^{\boxtimes d_k}\right)$ with multiplicity $\xi_1 \xi_2 \dots \xi_m$.

Then $\text{Ind} \left(\bigotimes_{k=1}^m M_k^{\boxtimes d_k} \right)$ is irreducible.

Proof. Let $N_k = \text{Ind} \left(M_k^{\boxtimes d_k} \right)$ for $k = 1, \dots, m$, and let Q be a nonzero quotient of the module $\text{Ind} \left(\bigotimes_{k=1}^m N_k \right)$. By (ii) and Lemma 1.5 (2), we have an injective homomorphism

$$\bigotimes_{k=1}^m N_k \hookrightarrow \text{Res}_{d_1\beta_1, \dots, d_m\beta_m} Q.$$

Hence, by (iii), \mathbf{i} occurs in $\text{ch}Q$ with multiplicity $\xi_1 \xi_2 \cdots \xi_m$. Since Q is arbitrary, we obtain that $\text{hd} \text{Ind} \left(\bigotimes_{k=1}^m N_k \right)$ is irreducible.

Now it follows from (i) that

$$\text{Ind} \left(\bigotimes_{k=1}^m N_k \right) \simeq \text{Ind} (N_m \boxtimes N_{m-1} \boxtimes \cdots \boxtimes N_1).$$

Let L be a nonzero submodule of $\text{Ind} \left(\bigotimes_{k=1}^m N_k \right)$. By (ii) and Lemma 1.5, there exists a surjective homomorphism

$$\text{Res}L \twoheadrightarrow \bigotimes_{k=1}^m N_k,$$

which implies that \mathbf{i} occurs in $\text{ch}L$ with multiplicity $\xi_1 \xi_2 \cdots \xi_m$ by (iii). Therefore, we conclude $\text{Ind} \left(\bigotimes_{k=1}^m N_k \right) = \text{Ind} \left(\bigotimes_{k=1}^m M_k^{\boxtimes d_k} \right)$ is irreducible. \square

Lemma 1.8. Let $\emptyset = I_{(0)} \subsetneq I_{(1)} \subsetneq I_{(2)} \subsetneq \cdots \subsetneq I_{(m)} \subset I$, and assume $\beta_k \in \sum_{i \in I_{(k)}} \mathbb{Z}_{\geq 0} \alpha_i$ for $k = 1, \dots, m$. Let $M_k \in R(\beta_k)\text{-fmod}$ ($1 \leq k \leq m$) be such that

- (i) $\varepsilon_i(M_k) = 0$ for $i \in I_{(k-1)}$,
- (ii) $\text{hd}M_k$ is irreducible, and
- (iii) $\text{hd}M_k$ occurs with multiplicity one as a composition factor of M_k .

Then

- (1) $\text{hd} \text{Ind} \left(\bigotimes_{k=1}^m M_k \right)$ is irreducible,
- (2) $\text{hd} \text{Ind} \left(\bigotimes_{k=1}^m M_k \right)$ occurs with multiplicity one as a composition factor of $\text{Ind} \left(\bigotimes_{k=1}^m M_k \right)$,
- (3) if $\tilde{f}_{\mathbf{i}_k} \mathbf{1} \simeq \text{hd}M_k$ for some $\mathbf{i}_k \in I^{\beta_k}$, then $\tilde{f}_{\mathbf{i}_1} \tilde{f}_{\mathbf{i}_2} \cdots \tilde{f}_{\mathbf{i}_m} \mathbf{1} \simeq \text{hd} \text{Ind} \left(\bigotimes_{k=1}^m M_k \right)$.

Proof. We may assume that $M_k \neq \mathbf{1}$ for $k = 1, \dots, m$. Let $d_k = |\beta_k|$ for $k = 1, \dots, m$, and set $d = d_1 + \cdots + d_m$. For $w \in S_d$ and a reduced expression $w = \sigma_{i_1} \cdots \sigma_{i_s}$, let

$$\tau_w = \tau_{i_1} \cdots \tau_{i_s}.$$

For $k = 1, \dots, m$, take $\mathbf{i}_k = (i_{k,1}, \dots, i_{k,d_k}) \in I^{\beta_k}$ such that $e(\mathbf{i}_k)M_k \neq 0$. It follows from (i) that $i_{k,1} \in I_{(k)} \setminus I_{(k-1)}$. Since $M_k \in R(\beta_k)\text{-fmod}$, we have

$$e(\beta_1, \dots, \beta_m)\tau_w e(\mathbf{i}_1 * \dots * \mathbf{i}_m) \neq 0 \quad \text{if and only if } w = \text{id}$$

for $w \in S_d/S_{d_1} \times \dots \times S_{d_m}$. Therefore,

$$\text{Res}_{\beta_1, \dots, \beta_m} \left(\text{Ind} \left(\bigboxtimes_{k=1}^m M_k \right) \right) \simeq \bigboxtimes_{k=1}^m M_k.$$

Let $L = \bigboxtimes_{k=1}^m \text{hd} M_k$. It follows from (ii) and (iii) that $[L]$ occurs with multiplicity one in $[\text{Res}_{\beta_1, \dots, \beta_m} \left(\text{Ind} \left(\bigboxtimes_{k=1}^m M_k \right) \right)]$ in the Grothendieck group $G_0(R(\beta_1) \otimes \dots \otimes R(\beta_m))$. Let Q be a nonzero quotient of $\text{Ind} \left(\bigboxtimes_{k=1}^m M_k \right)$. By Lemma 1.5, we have a nontrivial homomorphism

$$\bigboxtimes_{k=1}^m M_k \rightarrow \text{Res}_{\beta_1, \dots, \beta_m} Q,$$

which implies that $[L]$ occurs in $[\text{Res}_{\beta_1, \dots, \beta_m} Q]$ with multiplicity one. Since Q is arbitrary, we obtain assertions (1) and (2).

Now assume that $N = \text{hd} \text{Ind} \left(\bigboxtimes_{k=1}^m M_k \right)$, and let $N_k = \text{hd} M_k$ for $k = 1, \dots, m$. Then there is a surjective homomorphism

$$\text{Ind} \left(\bigboxtimes_{k=1}^m M_k \right) \twoheadrightarrow \text{Ind} \left(\bigboxtimes_{k=1}^m N_k \right),$$

which implies that $N = \text{hd} \text{Ind} \left(\bigboxtimes_{k=1}^m N_k \right)$. We will use induction on $|\text{wt}(N)|$. If $|\text{wt}(N)| = 0$, there is nothing to prove, so suppose that $|\text{wt}(N)| > 0$. Since N_1 is nontrivial, we can take $i \in I_{(1)}$ such that $\varepsilon_i(N_1) \neq 0$. Let $\varepsilon = \varepsilon_i(N_1)$. Then, by (i) and Lemma 1.5, we have

$$\varepsilon = \varepsilon_i \left(\text{Ind} \left(\bigboxtimes_{k=1}^m N_k \right) \right) = \varepsilon_i(N).$$

Since e_i is exact, there is a surjective homomorphism

$$e_i^\varepsilon \left(\text{Ind} \left(\bigboxtimes_{k=1}^m N_k \right) \right) \twoheadrightarrow e_i^\varepsilon N.$$

Then, by [14, Lem. 3.9], we know that $[e_i^\varepsilon N] = q_i^{-\varepsilon+1} [\varepsilon]_{q_i}! [\tilde{e}_i^\varepsilon N]$ and

$$\begin{aligned} \left[e_i^\varepsilon \left(\text{Ind} \left(\bigboxtimes_{k=1}^m N_k \right) \right) \right] &= [\text{Ind} ((e_i^\varepsilon N_1) \boxtimes N_2 \boxtimes \dots \boxtimes N_m)] \\ &= q_i^{-\varepsilon+1} [\varepsilon]_{q_i}! [\text{Ind} ((\tilde{e}_i^\varepsilon N_1) \boxtimes N_2 \boxtimes \dots \boxtimes N_m)] \end{aligned}$$

at the level of the Grothendieck group $G_0(R(\beta_1 + \dots + \beta_m - \varepsilon\alpha_i))$. Since $\text{hd} \text{Ind}((\tilde{e}_i^\varepsilon N_1) \boxtimes N_2 \boxtimes \dots \boxtimes N_m)$ is irreducible by (1), we obtain

$$\text{hd} \text{Ind}(\tilde{e}_i^\varepsilon N_1 \boxtimes \dots \boxtimes N_m) \simeq \tilde{e}_i^\varepsilon N.$$

Therefore, assertion (3) follows from a standard induction argument. \square

Let $n = |I|$ be the rank of $U_q(\mathfrak{g})$, and let $I_{(k)} \subset I$ ($k = 1, \dots, n$) be subsets of $I = I_{(n+1)}$ such that $I_{(k)} \subset I_{(k+1)}$ and $|I_{(k)}| = k$ for all k . We denote by $U_q(\mathfrak{g}_k)$ the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i ($i \in I_{(k)}$) and q^h ($h \in \mathbb{P}^\vee$). Let \mathfrak{B}_k be the crystal obtained from $B(\infty)$ by forgetting the i -arrows for $i \notin I_{(k)}$. Then \mathfrak{B}_k can be viewed as a $U_q(\mathfrak{g}_k)$ -crystal.

Lemma 1.9. *For each $k = 1, \dots, n$, every connected component of \mathfrak{B}_k has a unique highest weight vector.*

Proof. Let \mathcal{C} be a connected component of \mathfrak{B}_k . For $\lambda \in \mathbb{P}^+$, let $\iota_\lambda : B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \otimes C$ be the embedding in (1.1) and let

$$\tilde{\mathcal{C}} = \{v \otimes t_\lambda \otimes c \mid v \in \mathcal{C}\} \subset B(\infty) \otimes T_\lambda \otimes C.$$

By taking $\lambda \gg 0$, we can assume that $\tilde{\mathcal{C}} \cap \text{im}(\iota_\lambda) \neq \emptyset$. Since $\mathcal{C}_\lambda := \iota_\lambda^{-1}(\tilde{\mathcal{C}})$ is a nontrivial highest weight subcrystal of the crystal obtained from $B(\lambda)$ by forgetting the i -arrows with $i \notin I_{(k)}$, there is an element $u_{\mathcal{C}} \in \mathcal{C}$ such that $\iota_\lambda^{-1}(u_{\mathcal{C}} \otimes t_\lambda \otimes c)$ is the highest weight vector of the subcrystal \mathcal{C}_λ . Note that $u_{\mathcal{C}}$ does not depend on the choice of λ if $\tilde{\mathcal{C}} \cap \text{im}(\iota_\lambda) \neq \emptyset$. By construction, the element $u_{\mathcal{C}}$ is a unique highest weight vector of \mathcal{C} . \square

Take $v \in B(\infty)$. Let $u_0 = v$ and let u_k be the highest weight vector of the connected component \mathcal{C}_k of \mathfrak{B}_k containing v for $k = 1, \dots, n$. By construction, there is a chain of injective maps

$$\mathcal{C}_1 \hookrightarrow \mathcal{C}_2 \hookrightarrow \dots \hookrightarrow \mathcal{C}_{n-1} \hookrightarrow B(\infty).$$

For $k = 1, \dots, n$, let \mathbf{i}_k be a sequence of I such that $u_{k-1} = \tilde{f}_{\mathbf{i}_k} u_k$. Note that $v = \tilde{f}_{\mathbf{i}_1} \tilde{f}_{\mathbf{i}_1} \dots \tilde{f}_{\mathbf{i}_n} \mathbf{1}$. Let

$$(1.4) \quad \mathcal{N}_k(v) = \tilde{f}_{\mathbf{i}_k} \mathbf{1} \in \mathbb{B}(\infty).$$

Hence, for each $v \in B(\infty)$, we have the corresponding n -tuple $(\mathcal{N}_1(v), \mathcal{N}_2(v), \dots, \mathcal{N}_n(v))$ of modules in $\mathbb{B}(\infty)$.

Proposition 1.10.

- (1) *For $v \in B(\infty)$, let $\mathcal{N}_k(v)$ be the irreducible module defined by (1.4). Then $\text{hd Ind} \left(\bigboxtimes_{k=1}^n \mathcal{N}_k(v) \right)$ is irreducible.*
- (2) *The map $\Phi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ defined by*

$$\Phi(v) = \text{hd Ind} \left(\bigboxtimes_{k=1}^n \mathcal{N}_k(v) \right) \quad \text{for } v \in B(\infty)$$

is a crystal isomorphism.

Proof. Since it is obvious that $\Phi(1) = \mathbf{1}$, we assume $1 \neq v \in B(\infty)$ and let $\mathcal{N}_k(v) = \tilde{f}_{\mathbf{i}_k} \mathbf{1}$ be the corresponding irreducible module given in (1.4) for $k = 1, \dots, n$. By construction, we have

$$v = \tilde{f}_{\mathbf{i}_1} \tilde{f}_{\mathbf{i}_1} \cdots \tilde{f}_{\mathbf{i}_n} \mathbf{1} \quad \text{and} \quad \varepsilon_i(\mathcal{N}_k(v)) = 0 \quad \text{for all } i \in I_{(k-1)}.$$

Therefore, it follows from Lemma 1.8 that $\text{hd lnd} \left(\bigotimes_{k=1}^n \mathcal{N}_k(v) \right)$ is irreducible and

$$\Phi(\tilde{f}_{\mathbf{i}_1} \tilde{f}_{\mathbf{i}_2} \cdots \tilde{f}_{\mathbf{i}_n} \mathbf{1}) = \text{hd lnd} \left(\bigotimes_{k=1}^n \mathcal{N}_k(v) \right) = \tilde{f}_{\mathbf{i}_1} \tilde{f}_{\mathbf{i}_2} \cdots \tilde{f}_{\mathbf{i}_n} \Phi(1).$$

□

2. DESCRIPTION OF $\mathcal{N}_k(v)$ FOR FINITE TYPE VIA ADAPTED STRINGS

In this section, we give an explicit description of the modules $\mathcal{N}_k(v)$ for finite type. To that end, we choose a particular reduced expression of the longest element w_0 of the Weyl group W . Using the notion of adapted strings for crystals given in [23] with respect to these specially chosen expressions, we describe $\mathcal{N}_k(v)$ explicitly in terms of Kashiwara operators \tilde{f}_i (Proposition 2.3).

Throughout the section, we assume that the Cartan datum $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ is of finite type. For a sequence $\mathbf{m} = (m_1, \dots, m_k)$ of elements in I , we write $r_{\mathbf{m}} = r_{m_1} r_{m_2} \cdots r_{m_k}$ for the corresponding element in the Weyl group. Using this convention, we fix a particular reduced expression for the longest element w_0 of W

$$(2.1) \quad w_0 = r_{\mathbf{s}_1} r_{\mathbf{s}_2} \cdots r_{\mathbf{s}_n} = r_{s_1} r_{s_2} \cdots r_{s_\ell}$$

where the sequences \mathbf{s}_k are displayed in Table 1 below. Let l_k be the length of the sequence \mathbf{s}_k in (2.1) for $k = 1, \dots, n$. Then $r_{\mathbf{s}_1} = r_{s_1} \cdots r_{s_{l_1}}$, $r_{\mathbf{s}_2} = r_{s_{l_1+1}} \cdots r_{s_{l_1+l_2}}$ and so forth. Note that ℓ is the length of the reduced expression.

Definition 2.1. Let $v \in B(\infty)$ (resp. $v \in B(\lambda)$). An ℓ -tuple $\mathbf{a}(v) = (a_1, \dots, a_\ell) \in (\mathbb{Z}_{\geq 0})^\ell$ is called the *adapted string* of v with respect to the expression $r_{s_1} \cdots r_{s_\ell}$ if

$$a_1 = \varepsilon_{s_1}(v), \quad a_2 = \varepsilon_{s_2}(\tilde{e}_{s_1}^{a_1} v), \quad \dots, \quad a_\ell = \varepsilon_{s_\ell}(\tilde{e}_{s_{\ell-1}}^{a_{\ell-1}} \cdots \tilde{e}_{s_1}^{a_1} v).$$

We denote by $\mathcal{S} = \{\mathbf{a}(v) \mid v \in B(\infty)\}$ (resp. $\mathcal{S}^\lambda = \{\mathbf{a}(v) \mid v \in B(\lambda)\}$) the set of all adapted strings of elements in $B(\infty)$ (resp. $B(\lambda)$).

Observe that if $\iota_\lambda(v) = v' \otimes t_\lambda \otimes c$ for some $v \in B(\lambda)$ and $v' \in B(\infty)$, then $\mathbf{a}(v) = \mathbf{a}(v')$. It follows from [16, 22] that \mathcal{S} and \mathcal{S}^λ are in one-to-one correspondence with the crystals $B(\infty)$

TABLE 1.

Dynkin diagrams		$w_0 = r_{s_1} \cdots r_{s_n}$
(A _n)		$s_k = (n+1-k, \dots, n-1, n)$ for $k = 1, \dots, n$
(B _n)		$s_k = (n+1-k, \dots, n-1, n, n-1, \dots, n+1-k)$ for $k = 1, \dots, n$
(C _n)		s_k is the same as for type B _n for $k = 1, \dots, n$
(D _n)		$s_1 = (n), s_2 = (n-1),$ $s_k = (n+1-k, \dots, n-2, n, n-1, n-2, \dots, n+1-k)$ for $k = 3, \dots, n$
(E ₆)		s_k is the same as for type D ₅ for $k = 1, \dots, 5,$ $s_6 = (6, 5, 3, 4, 2, 1, 3, 2, 5, 3, 4, 6, 5, 3, 2, 1),$
(E ₇)		s_k is the same as for type E ₆ for $k = 1, \dots, 6,$ $s_7 = (7, 6, 5, 3, 4, 2, 1, 3, 2, 5, 3, 4, 6, 5, 3, 2, 1, 7, 6, 5, 3, 4, 2, 3, 5, 6, 7)$
(E ₈)		s_k is the same as for type E ₇ for $k = 1, \dots, 7,$ $s_8 = (8, 7, 6, 5, 3, 4, 2, 1, 3, 2, 5, 3, 4, 6, 5, 3, 2, 1, 7, 6, 5, 3, 4, 2, 3, 5, 6, 7, 8, 7, 6, 5, 3, 4, 2, 1, 3, 2, 5, 3, 4, 6, 5, 3, 2, 1, 7, 6, 5, 3, 4, 2, 3, 5, 6, 7, 8)$
(F ₄)		s_k is the same as for type B ₃ for $k = 1, \dots, 3,$ $s_4 = (4, 3, 2, 1, 3, 2, 3, 4, 3, 2, 1, 3, 2, 3, 4)$
(G ₂)		$s_1 = (1), s_2 = (2, 1, 2, 1, 2)$

and $B(\lambda)$, respectively, by the maps

$$\begin{aligned}
 (2.2) \quad \mathcal{S} &\xrightarrow{1-1} B(\infty) \quad \text{given by} \quad \mathbf{a} = (a_1, \dots, a_\ell) \mapsto \tilde{f}_{s_1}^{a_1} \cdots \tilde{f}_{s_\ell}^{a_\ell} 1, \\
 \mathcal{S}^\lambda &\xrightarrow{1-1} B(\lambda) \quad \text{given by} \quad \mathbf{a} = (a_1, \dots, a_\ell) \mapsto \tilde{f}_{s_1}^{a_1} \cdots \tilde{f}_{s_\ell}^{a_\ell} b_\lambda.
 \end{aligned}$$

For $\mathbf{a} = (a_1, \dots, a_\ell) \in (\mathbb{Z}_{\geq 0})^\ell$, set

$$(2.3) \quad \mathbf{a}_{k,j} = a_{l_1+\dots+l_{k-1}+j} \quad \text{and} \quad \mathbf{a}_k = (\mathbf{a}_{k,1}, \mathbf{a}_{k,2}, \dots, \mathbf{a}_{k,l_k}),$$

where l_k is, as above, the length of \mathbf{s}_k . Then $\mathbf{a} = \mathbf{a}_1 * \dots * \mathbf{a}_n$.

In particular, for $v \in B(\infty)$, we have $\mathbf{a}(v) = \mathbf{a}(v)_1 * \dots * \mathbf{a}(v)_n$ and

$$v = \tilde{f}_{\mathbf{s}_1}^{\mathbf{a}(v)_1} \tilde{f}_{\mathbf{s}_2}^{\mathbf{a}(v)_2} \dots \tilde{f}_{\mathbf{s}_n}^{\mathbf{a}(v)_n} \mathbf{1}.$$

Proposition 2.2 ([23]). *Let $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ be a Cartan datum of finite type. Then \mathcal{S} has the following description:*

- (A_n) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{i,1} \geq \mathbf{a}_{i,2} \geq \dots \geq \mathbf{a}_{i,i} \ (1 \leq i \leq n)\}$
- (B_n) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid 2\mathbf{a}_{i,1} \geq \dots \geq 2\mathbf{a}_{i,i-1} \geq \mathbf{a}_{i,i} \geq 2\mathbf{a}_{i,i+1} \geq \dots \geq 2\mathbf{a}_{i,2i-1} \ (1 \leq i \leq n)\}$
- (C_n) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{i,1} \geq \dots \geq \mathbf{a}_{i,i-1} \geq \mathbf{a}_{i,i} \geq \mathbf{a}_{i,i+1} \geq \dots \geq \mathbf{a}_{i,2i-1} \ (1 \leq i \leq n)\}$
- (D_n) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{i,1} \geq \dots \geq \mathbf{a}_{i,i-1}, \mathbf{a}_{i,i} \geq \mathbf{a}_{i,i+1} \geq \dots \geq \mathbf{a}_{i,2i-2} \ (3 \leq i \leq n)\}$
- (E₆) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{6,1} \geq \mathbf{a}_{6,2} \geq \mathbf{a}_{6,3} \geq \mathbf{a}_{6,4}, \mathbf{a}_{6,5} \geq \mathbf{a}_{6,7} \geq \mathbf{a}_{6,8}, \mathbf{a}_{6,9} \geq \mathbf{a}_{6,10} \geq \mathbf{a}_{6,11}, \mathbf{a}_{6,13}$
 $\geq \mathbf{a}_{6,14} \geq \mathbf{a}_{6,15} \geq \mathbf{a}_{6,16}; \mathbf{a}_{6,5} \geq \mathbf{a}_{6,6} \geq \mathbf{a}_{6,8}; \mathbf{a}_{6,9} \geq \mathbf{a}_{6,12} \geq \mathbf{a}_{6,13};$
 $\mathbf{a}_{i,j} \text{'s satisfy the inequalities for } D_5 \text{ for } i = 1, \dots, 5\}$
- (E₇) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{7,1} \geq \mathbf{a}_{7,2} \geq \mathbf{a}_{7,3} \geq \mathbf{a}_{7,4} \geq \mathbf{a}_{7,5}, \mathbf{a}_{7,6} \geq \mathbf{a}_{7,8} \geq \mathbf{a}_{7,9}, \mathbf{a}_{7,10}$
 $\geq \mathbf{a}_{7,11} \geq \mathbf{a}_{7,12}, \mathbf{a}_{7,14} \geq \mathbf{a}_{7,15} \geq \mathbf{a}_{7,16}, \mathbf{a}_{7,20} \geq \mathbf{a}_{7,21} \geq \mathbf{a}_{7,22}, \mathbf{a}_{7,23}$
 $\geq \mathbf{a}_{7,24} \geq \mathbf{a}_{7,25} \geq \mathbf{a}_{7,26} \geq \mathbf{a}_{7,27}; \mathbf{a}_{7,6} \geq \mathbf{a}_{7,7} \geq \mathbf{a}_{7,8};$
 $\mathbf{a}_{7,10} \geq \mathbf{a}_{7,13} \geq \mathbf{a}_{7,14}, \mathbf{a}_{7,18} \geq \mathbf{a}_{7,19} \geq \mathbf{a}_{7,20}; \mathbf{a}_{7,16} \geq \mathbf{a}_{7,17} \geq \mathbf{a}_{7,23};$
 $\mathbf{a}_{i,j} \text{'s satisfy the inequalities for } E_6 \text{ for } i = 1, \dots, 6\}$
- (E₈) : $\{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid (\mathbf{a}_{8,2}, \dots, \mathbf{a}_{8,28}) \text{ and } (\mathbf{a}_{8,30}, \dots, \mathbf{a}_{8,56}) \text{ satisfy the inequalities of } \mathbf{a}_7 \text{ for } E_7;$
 $\mathbf{a}_{8,1} \geq \mathbf{a}_{8,2}; \mathbf{a}_{8,29} \geq \mathbf{a}_{8,30}; \mathbf{a}_{8,56} \geq \mathbf{a}_{8,57};$
 $\mathbf{a}_{8,19} \geq \max\{\mathbf{a}_{8,30}, \mathbf{a}_{8,29} - \mathbf{a}_{8,28}\}; \min\{\mathbf{a}_{8,23}, \mathbf{a}_{8,25} + \mathbf{a}_{8,33} - \mathbf{a}_{8,34}\} \geq \mathbf{a}_{8,35};$
 $\mathbf{a}_{8,20} \geq \max\{\mathbf{a}_{8,31}, \mathbf{a}_{8,28} + \mathbf{a}_{8,30} - \mathbf{a}_{8,27}\}; \min\{\mathbf{a}_{8,25}, \mathbf{a}_{8,26} + \mathbf{a}_{8,32} - \mathbf{a}_{8,33}\} \geq \mathbf{a}_{8,37};$
 $\mathbf{a}_{8,21} \geq \max\{\mathbf{a}_{8,32}, \mathbf{a}_{8,27} + \mathbf{a}_{8,31} - \mathbf{a}_{8,26}\}; \min\{\mathbf{a}_{8,26}, \mathbf{a}_{8,27} + \mathbf{a}_{8,31} - \mathbf{a}_{8,32}\} \geq \mathbf{a}_{8,39};$
 $\mathbf{a}_{8,22} \geq \max\{\mathbf{a}_{8,33}, \mathbf{a}_{8,26} + \mathbf{a}_{8,32} - \mathbf{a}_{8,25}\}; \min\{\mathbf{a}_{8,27}, \mathbf{a}_{8,28} + \mathbf{a}_{8,30} - \mathbf{a}_{8,31}\} \geq \mathbf{a}_{8,42};$
 $\mathbf{a}_{8,24} \geq \max\{\mathbf{a}_{8,34}, \mathbf{a}_{8,25} + \mathbf{a}_{8,33} - \mathbf{a}_{8,23}\}; \min\{\mathbf{a}_{8,28}, \mathbf{a}_{8,29} - \mathbf{a}_{8,30}\} \geq \mathbf{a}_{8,47};$
 $\mathbf{a}_{i,j} \text{'s satisfy the inequalities for } E_7 \text{ for } i = 1, \dots, 7\}$

$$\begin{aligned}
 (\text{F}_4) : \quad & \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid \mathbf{a}_{4,1} \geq \mathbf{a}_{4,2} \geq \mathbf{a}_{4,3} \geq \mathbf{a}_{4,4}, \mathbf{a}_{4,5} \geq \mathbf{a}_{4,6} \geq \mathbf{a}_{4,7}; \\
 & \mathbf{a}_{4,9} \geq \mathbf{a}_{4,10} \geq \mathbf{a}_{4,11}, \mathbf{a}_{4,12} \geq \mathbf{a}_{4,13} \geq \mathbf{a}_{4,14} \geq \mathbf{a}_{4,15}; \\
 & \mathbf{a}_{4,5} \geq \mathbf{a}_{4,9}; \mathbf{a}_{4,7} \geq \mathbf{a}_{4,12}; \mathbf{a}_{4,5} + \mathbf{a}_{4,7} \geq \mathbf{a}_{4,8} \geq \mathbf{a}_{4,9} + \mathbf{a}_{4,12}; \\
 & 2\mathbf{a}_{4,6} \geq \mathbf{a}_{4,7} + \mathbf{a}_{4,9} \geq 2\mathbf{a}_{4,10}; \\
 & \mathbf{a}_{i,j} \text{'s satisfy the inequalities for } \mathbf{B}_3 \text{ for } i = 1, 2, 3\} \\
 (\text{G}_2) : \quad & \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid 6\mathbf{a}_{2,1} \geq 2\mathbf{a}_{2,2} \geq 3\mathbf{a}_{2,3} \geq 2\mathbf{a}_{2,4} \geq 6\mathbf{a}_{2,5}\}.
 \end{aligned}$$

We now give a description of $\mathcal{N}_k(v)$ using Proposition 2.2.

Proposition 2.3. *Let $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ be a Cartan datum of finite type. For $v \in B(\infty)$ and $k = 1, \dots, n$, the module $\mathcal{N}_k(v)$ defined by (1.4) is given by*

$$\mathcal{N}_k(v) = \tilde{f}_{\mathbf{s}_k}^{\mathbf{a}(v)} \mathbf{1},$$

where $\mathbf{a}(v)$ is the adapted string of v with respect to the expression $w_0 = r_{\mathbf{s}_1} \cdots r_{\mathbf{s}_n}$ in Table 1.

Proof. Let $I_{(k)} = \{n+1-k, \dots, n-1, n\}$ for $k = 1, \dots, n$ and set $I_{(0)} = \emptyset$. Let \mathfrak{B}_k denote the crystal obtained from $B(\infty)$ by forgetting the i -arrows with $i \notin I_{(k)}$. Take $v \in B(\infty)$ and let $\mathbf{a} = \mathbf{a}(v) = \mathbf{a}_1 * \cdots * \mathbf{a}_n$ be the adapted string of v with respect to the expression $w_0 = r_{\mathbf{s}_1} \cdots r_{\mathbf{s}_n}$ in Table 1. For $k = 1, \dots, n$, let $\mathbf{0}_k = \underbrace{(0, \dots, 0)}_{l_k}$ where l_k is the length of \mathbf{s}_k as before. Then, by Proposition 2.2,

$$\mathbf{b}_k := \mathbf{0}_1 * \cdots * \mathbf{0}_k * \mathbf{a}_{k+1} * \cdots * \mathbf{a}_{n-1} * \mathbf{a}_n$$

is contained in \mathcal{S} . Let $u_k = \tilde{f}_{\mathbf{s}_{k+1}}^{\mathbf{a}_{k+1}} \cdots \tilde{f}_{\mathbf{s}_n}^{\mathbf{a}_n} \mathbf{1}$ for $k = 0, \dots, n-1$ and let $u_n = \mathbf{1}$. Since \mathbf{b}_k is the adapted string of u_k , by the definition of adapted strings we have

$$v = \tilde{f}_{\mathbf{s}_1}^{\mathbf{a}_1} \cdots \tilde{f}_{\mathbf{s}_k}^{\mathbf{a}_k} u_k, \quad \varepsilon_i(u_k) = 0 \text{ for } i \in I_{(k)},$$

which implies that u_k is the highest weight vector of the connected component \mathcal{C}_k of \mathfrak{B}_k containing v . Therefore, $\mathcal{N}_k(v) = \tilde{f}_{\mathbf{s}_k}^{\mathbf{a}_k} \mathbf{1}$ for $k = 1, \dots, n$. \square

3. EXPLICIT CONSTRUCTION OF IRREDUCIBLE MODULES FOR FINITE CLASSICAL TYPE

In Sections 3 and 4, we assume that \mathfrak{A} is of finite classical type. We maintain the notation from Sections 1 and 2. Our aim in this section is to present an explicit construction of irreducible R -modules (resp. R^λ -modules) for finite classical type using a single induction step. Let \mathbf{B} be the crystal of the irreducible highest weight $U_q(\mathfrak{g})$ -module $V(\Lambda_n)$ if \mathfrak{A} is of type A_n and of $V(\Lambda_1)$ if \mathfrak{A} is of type B_n, C_n, D_n . We define the 1 or 2-dimensional irreducible

R -module $\nabla_{(a,b)}$ for $a, b \in \mathbf{B}$ with $a \succ b$ by using the structure of \mathbf{B} . Combining Proposition 1.10 with some facts about the modules $\nabla_{(a,b)}$ and the descriptions of the adapted strings, we define the outer tensor product $\nabla(\mathbf{a}(v)) := \nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)$ of $\nabla_{(a,b)}$'s for $v \in B(\infty)$ (resp. $v \in B(\lambda)$). For $v \in B(\lambda)$, the number $\eta(v)$ of $\nabla_{(a,b)}$'s in $\nabla(\mathbf{a}(v))$ has an upper bound; i.e., $\eta(v) \leq n\lambda(h)$, where h is as in Lemma 3.4 below. Then, we construct a map $\Psi : B(\infty) \rightarrow \mathbb{B}(\infty)$ (resp. $\Psi^\lambda : B(\lambda) \rightarrow \mathbb{B}(\lambda)$) by taking the head of $\text{Ind} \nabla(\mathbf{a}(v))$ (Theorem 3.2). The proof that this map is indeed a crystal isomorphism, hence is compatible with the Kashiwara operators, will be provided in the next section.

We first give a detailed description of \mathcal{S} and \mathcal{S}^λ for finite classical type with respect to the expression of $w_0 = r_{s_1} \cdots r_{s_n}$ in Table 1. Let \triangle_{A_n} be the triangle consisting of right justified rows of boxes with 1 box in the first (bottom) row, 2 boxes in the second row, \dots , and n boxes in the top row. Let \triangle_{B_n} (resp. \triangle_{D_n}) be the triangle consisting of centered rows of boxes having 1 box (resp. 2 boxes) in the first row, 3 boxes (resp. 4 boxes) in the second row, \dots , and $(2n-1)$ boxes (resp. $(2n-2)$ boxes) in the top row. Set $\triangle_{C_n} = \triangle_{B_n}$. When it is not necessary to specify the type, we omit the subscript and simply write \triangle . For $\mathbf{a} \in \mathcal{S}$, let $\triangle(\mathbf{a})$ denote the filling of the triangle \triangle with entries of \mathbf{a} from left to right in each row, and from bottom to top. Let t_{ij} be the j th entry of the i th row in $\triangle(\mathbf{a})$ and let $\triangle(\mathbf{a}) = \{t_{ij}\}_{1 \leq i \leq n', p_i \leq j \leq p'_i}$, where

$$\begin{aligned} n' &= \begin{cases} n & (A_n, B_n, C_n) \\ n-1 & (D_n) \end{cases} \\ p_i &= \begin{cases} n+1-i & (A_n, B_n, C_n), \\ n-i & (D_n) \end{cases} \\ p'_i &= \begin{cases} n & (A_n), \\ n-1+i & (B_n, C_n, D_n). \end{cases} \end{aligned}$$

Set $t_{ij} = 0$ except when $1 \leq i \leq n'$, $p_i \leq j \leq p'_i$. For example, if $\mathbf{a} = (2, 3, 1, 0, 9, 8, 4, 2, 1) \in \mathcal{S}$ for type B_3 and $\mathbf{a}' = (5, 2, 7, 4, 3, 1, 9, 6, 4, 5, 3, 2) \in \mathcal{S}$ for type D_4 , then $t_{1,3} = 2$ in $\triangle(\mathbf{a})$, $t_{3,4} = 5$ in $\triangle(\mathbf{a}')$ and

$$\triangle(\mathbf{a}) = \begin{array}{ccccc} \boxed{9} & \boxed{8} & \boxed{4} & \boxed{2} & \boxed{1} \\ & \boxed{3} & \boxed{1} & \boxed{0} & \\ & & \boxed{2} & & \end{array}, \quad \triangle(\mathbf{a}') = \begin{array}{cccccc} \boxed{9} & \boxed{6} & \boxed{4} & \boxed{5} & \boxed{3} & \boxed{2} \\ & \boxed{7} & \boxed{4} & \boxed{3} & \boxed{1} & \\ & & \boxed{5} & \boxed{2} & & \end{array}.$$

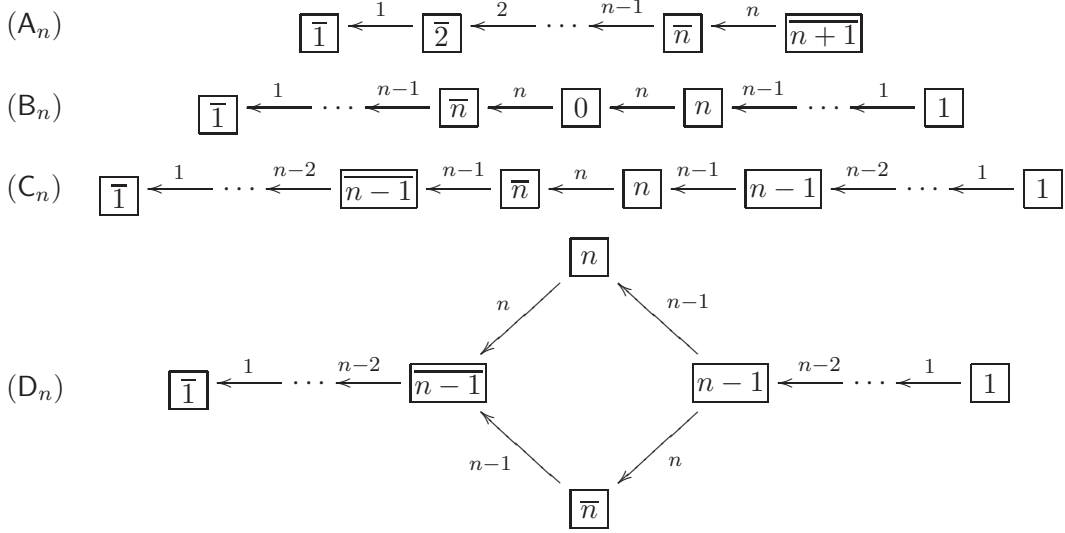
Let $\tilde{c}(t_{i,n-1}) = \sum_{k=i}^{n-1} t_{k,n-1}$, $\tilde{c}(t_{i,n}) = \sum_{k=i}^{n-1} t_{k,n}$, and

$$(3.1) \quad c(t_{i,j}) = \begin{cases} \sum_{k=i}^n t_{k,j} & \text{if } j \leq n \text{ (A}_n\text{), } j = n \text{ (B}_n\text{),} \\ \sum_{k=i}^n (t_{k,j} + t_{k,2n-j}) & \text{if } j < n \text{ (B}_n\text{, C}_n\text{),} \\ \sum_{k=i}^{n-1} (t_{k,j} + t_{k,2n-1-j}) & \text{if } j < n-1 \text{ (D}_n\text{),} \\ \sum_{k=i}^n 2t_{k,j} & \text{if } j = n \text{ (C}_n\text{),} \\ \sum_{k=i}^{n-1} (t_{k,n-1} + t_{k,n}) & \text{if } j = n-1, n \text{ (D}_n\text{),} \\ t_{i,j} + \sum_{k=i+1}^n (t_{k,2n-j} + t_{k,j}) & \text{if } j > n \text{ (B}_n\text{, C}_n\text{),} \\ t_{i,j} + \sum_{k=i+1}^{n-1} (t_{k,2n-1-j} + t_{k,j}) & \text{if } j > n \text{ (D}_n\text{).} \end{cases}$$

Proposition 3.1 ([23]). *Let $(\mathfrak{A}, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum of finite classical type, and let $\lambda = \lambda_1 \Lambda_1 + \dots + \lambda_n \Lambda_n \in P^+$. For $\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell$, let $\Delta(\mathbf{a}) = \{t_{ij}\}_{1 \leq i \leq n', p_i \leq j \leq p'_i}$ be as above and assume $c(t_{i,j})$ is as in (3.1). Then*

$$\begin{aligned} (\text{A}_n) : \mathcal{S} &= \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid t_{i,n+1-i} \geq t_{i,n+2-i} \geq \dots \geq t_{i,n} \text{ for } 1 \leq i \leq n\}, \\ \mathcal{S}^\lambda &= \{\mathbf{a} \in \mathcal{S} \mid t_{i,j} \leq \lambda_j + c(t_{i+1,j-1}) - 2c(t_{i+1,j}) + c(t_{i,j+1}) \\ &\quad \text{for } 1 \leq i \leq n, n+1-i \leq j \leq n\} \\ (\text{B}_n) : \mathcal{S} &= \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid 2t_{i,n+1-i} \geq \dots \geq 2t_{i,n-1} \geq t_{i,n} \geq 2t_{i,n+1} \geq \dots \geq 2t_{i,n-1+i} \\ &\quad \text{for } 1 \leq i \leq n\}, \\ \mathcal{S}^\lambda &= \{\mathbf{a} \in \mathcal{S} \mid t_{i,j} \leq \lambda_j + c(t_{i,j+1}) - 2c(t_{i,2n-j}) + c(t_{i,2n+1-j}), \\ &\quad t_{i,2n-j} \leq \lambda_j + c(t_{i+1,j+1}) - 2c(t_{i+1,j}) + c(t_{i,2n+1-j}), \\ &\quad t_{i,n} \leq \lambda_n + 2c(t_{i,n+1}) - 2c(t_{i+1,n}) \text{ for } 1 \leq i \leq n, n+1-i \leq j < n\} \\ (\text{C}_n) : \mathcal{S} &= \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid t_{i,n+1-i} \geq \dots \geq t_{i,n} \geq \dots \geq t_{i,n-1+i} \text{ for } 1 \leq i \leq n\} \\ \mathcal{S}^\lambda &= \{\mathbf{a} \in \mathcal{S} \mid t_{i,j} \leq \lambda_j + c(t_{i,j+1}) - 2c(t_{i,2n-j}) + c(t_{i,2n+1-j}), \\ &\quad t_{i,2n-j} \leq \lambda_j + c(t_{i+1,j+1}) - 2c(t_{i+1,j}) + c(t_{i,2n+1-j}), \\ &\quad t_{i,n} \leq \lambda_n + c(t_{i,n+1}) - c(t_{i+1,n}) \text{ for } 1 \leq i \leq n, n+1-i \leq j < n\} \\ (\text{D}_n) : \mathcal{S} &= \{\mathbf{a} \in (\mathbb{Z}_{\geq 0})^\ell \mid t_{i,n-i} \geq \dots \geq t_{i,n-2} \geq t_{i,n-1}, t_{i,n} \geq t_{i,n+1} \geq \dots \geq t_{i,n-1+i} \\ &\quad \text{for } 1 \leq i \leq n-1\}, \\ \mathcal{S}^\lambda &= \{\mathbf{a} \in \mathcal{S} \mid t_{i,j} \leq \lambda_j + c(t_{i,j+1}) - 2c(t_{i,2n-1-j}) + c(t_{i,2n-j}), \\ &\quad t_{i,2n-1-j} \leq \lambda_j + c(t_{i+1,j+1}) - 2c(t_{i+1,j}) + c(t_{i,2n-j}), \\ &\quad t_{i,n-1} \leq \lambda_{n-1} + c(t_{i,n+1}) - 2\tilde{c}(t_{i+1,n-1}), \\ &\quad t_{i,n} \leq \lambda_n + c(t_{i,n+1}) - 2\tilde{c}(t_{i+1,n}) \text{ for } 1 \leq i \leq n-1, n-i \leq j < n-1\}. \end{aligned}$$

Let \mathbf{B} be the crystal given by



with the entries ordered by

$$\begin{aligned}
(A_n) \quad & \overline{1} \succ \overline{2} \succ \cdots \succ \overline{n+1}, \\
(B_n) \quad & \overline{1} \succ \overline{2} \succ \cdots \succ \overline{n} \succ 0 \succ n \succ \cdots \succ 2 \succ 1, \\
(C_n) \quad & \overline{1} \succ \overline{2} \succ \cdots \succ \overline{n-1} \succ \overline{n} \succ n \succ n-1 \succ \cdots \succ 1, \\
(D_n) \quad & \overline{1} \succ \overline{2} \succ \cdots \succ \overline{n-1} \succ \overline{n}, n \succ n-1 \succ \cdots \succ 1.
\end{aligned}$$

Let

$$(3.2) \quad \widehat{i} = \begin{cases} \overline{i} & \text{if } 1 \leq i \leq n+1 \text{ (A}_n\text{), } 1 \leq i \leq n \text{ (B}_n\text{, C}_n\text{, D}_n\text{),} \\ 0 & \text{if } i = n+1 \text{ (B}_n\text{),} \\ 2n+1-i & \text{if } n+1 \leq i \leq 2n \text{ (C}_n\text{, D}_n\text{),} \\ 2n+2-i & \text{if } n+2 \leq i \leq 2n+1 \text{ (B}_n\text{).} \end{cases}$$

For $a, b \in \mathbf{B}$ with $a \succ b$, let $\mathbf{i}(a, b)$ be a sequence of elements in I such that $a = \tilde{f}_{\mathbf{i}(a, b)} b$, and define the irreducible graded R -module

$$(3.3) \quad \nabla_{(a, b)} = \tilde{f}_{\mathbf{i}(a, b)} \mathbf{1}.$$

Note that $\nabla_{(a, b)}$ in general is not a cuspidal representation given in [6, 20]. The action of the KLR algebra can be described explicitly as follows.

If one of the following holds: $a \succ b$ (A_n, C_n), either $b \succeq 0$ or $0 \succeq a$ (B_n), either $b \succ n-1$ or $\overline{n-1} \succ a$ (D_n), then the module $\nabla_{(a, b)}$ is the 1-dimensional R -module $\mathbb{C}v$ specified by

$$x_i v = 0, \quad \tau_j v = 0, \quad e(\mathbf{i})v = \begin{cases} v & \text{if } \mathbf{i} = \mathbf{i}(a, b), \\ 0 & \text{otherwise.} \end{cases}$$

If $a \succ 0 \succ b$ for type \mathbf{B}_n , then $\nabla_{(a,b)}$ is the 2-dimensional R -module $\mathbb{C}u \oplus \mathbb{C}v$ with R -action

$$\begin{aligned} x_i u &= 0, & \tau_j u &= \begin{cases} v & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} & e(\mathbf{i})u &= \begin{cases} u & \text{if } \mathbf{i} = \mathbf{i}(a, b), \\ 0 & \text{otherwise,} \end{cases} \\ x_i v &= \begin{cases} -u & \text{if } i = d, \\ u & \text{if } i = d + 1, \\ 0 & \text{otherwise} \end{cases} & \tau_j v &= 0, & e(\mathbf{i})v &= \begin{cases} v & \text{if } \mathbf{i} = \mathbf{i}(a, b), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where d is an integer such that $\sigma_d(\mathbf{i}(a, b)) = \mathbf{i}(a, b)$.

If $a \succeq \overline{n-1}$ and $n-1 \succeq b$ for type \mathbf{D}_n , then the module $\nabla_{(a,b)}$ is the 2-dimensional R -module $\mathbb{C}u \oplus \mathbb{C}v$ given by

$$\begin{aligned} x_i u &= 0, & \tau_j u &= \begin{cases} \mathcal{Q}_{n,n-1}(x_n, x_{n-1})v & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} & e(\mathbf{i})u &= \begin{cases} u & \text{if } \mathbf{i} = \mathbf{i}(a, n) * \mathbf{i}(n, b), \\ 0 & \text{otherwise,} \end{cases} \\ x_i v &= 0, & \tau_j v &= \begin{cases} u & \text{if } j = d, \\ 0 & \text{otherwise,} \end{cases} & e(\mathbf{i})v &= \begin{cases} v & \text{if } \mathbf{i} = \mathbf{i}(a, \overline{n}) * \mathbf{i}(\overline{n}, b), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where d is an integer such that $\sigma_d(\mathbf{i}(a, n) * \mathbf{i}(n, b)) = \mathbf{i}(a, \overline{n}) * \mathbf{i}(\overline{n}, b)$. Note that $\mathcal{Q}_{n,n-1}(x_n, x_{n-1}) = \zeta_{n,n-1} \in \mathbb{C} \setminus \{0\}$.

It follows from the description above that, for $a, b \in \mathbf{B}$ with $a \succ b$,

$$(3.4) \quad \text{ch} \nabla_{(a,b)} = \begin{cases} \mathbf{i}(a, n) * \mathbf{i}(n, b) + \mathbf{i}(a, \overline{n}) * \mathbf{i}(\overline{n}, b) & \text{if } a \succeq \overline{n-1}, n-1 \succeq b \text{ } (\mathbf{D}_n), \\ 2\mathbf{i}(a, b) & \text{if } a \succ 0 \succ b \text{ } (\mathbf{B}_n), \\ \mathbf{i}(a, b) & \text{otherwise.} \end{cases}$$

Remark. We see from the expression for the character in (3.4) that $\nabla_{(a,b)}$ can be identified with the sequence $\mathbf{i}(a, b)$, or equivalently, with the segment of \mathbf{B} between a and b , which is another reason why we defined the Kashiwara operators in the opposite manner to [14, 18].

Given $\mathbf{a} \in \mathcal{S}$ (resp. $\mathbf{a} \in \mathcal{S}^\lambda$) and $i = 1, \dots, n$, let

$$(3.5) \quad \nabla(\mathbf{a}; i) = \begin{cases} \bigotimes_{j=n+1-i}^{n_i} \left(\nabla_{(\widehat{n+1-i}, \widehat{j+1})}^{\boxtimes \vartheta_{ij}} \right) & \text{if } 1 \leq i \leq n \text{ } (\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n), \\ \nabla_{(\overline{n-1}, n)}^{\boxtimes t_{1,n-1}} & \text{if } i = 1 \text{ } (\mathbf{D}_n), \\ \nabla_{(\overline{n-1}, \overline{n})}^{\boxtimes t_{1,n}} & \text{if } i = 2 \text{ } (\mathbf{D}_n), \\ \bigotimes_{j=n+1-i}^{n_i} \left(\nabla_{(\widehat{n+1-i}, \widehat{j+1})}^{\boxtimes \vartheta_{i-1,j}} \right) & \text{if } 3 \leq i \leq n \text{ } (\mathbf{D}_n), \end{cases}$$

where $n_i = n$ for all i (\mathbf{A}_n), $n_i = n + i$ (\mathbf{B}_n), $n_i = n - 1 + i$ ($\mathbf{C}_n, \mathbf{D}_n$), $\triangle(\mathbf{a}) = \{t_{i,j}\}$, and

$$(3.6) \quad \vartheta_{ij} = \begin{cases} t_{i,j} - t_{i,j+1} & \text{if } j \leq n_i \text{ } (A_n, C_n), j \leq n-2 \text{ } (B_n), j \leq n-3 \text{ } (D_n), \\ t_{i,n-1} - \lceil \frac{t_{i,n}}{2} \rceil & \text{if } j = n-1 \text{ } (B_n), \\ \lceil \frac{t_{i,n}}{2} \rceil - \lfloor \frac{t_{i,n}}{2} \rfloor & \text{if } j = n \text{ } (B_n), \\ \lfloor \frac{t_{i,n}}{2} \rfloor - t_{i,n+1} & \text{if } j = n+1 \text{ } (B_n), \\ t_{i,n-2} - \max\{t_{i,n-1}, t_{i,n}\} & \text{if } j = n-2 \text{ } (D_n), \\ \max\{0, t_{i,n} - t_{i,n-1}\} & \text{if } j = n-1 \text{ } (D_n), \\ \max\{0, t_{i,n-1} - t_{i,n}\} & \text{if } j = n \text{ } (D_n), \\ \min\{t_{i,n-1}, t_{i,n}\} - t_{i,n+1} & \text{if } j = n+1 \text{ } (D_n), \\ t_{i,j-1} - t_{i,j} & \text{if } j \geq n+2 \text{ } (B_n, D_n). \end{cases}$$

For $v \in B(\infty)$ (resp. $v \in B(\lambda)$), let $\mathbf{a}(v)$ be the adapted string of v with respect to the expression $w_0 = r_{s_1} \cdots r_{s_n}$ given in Table 1. Now we are ready to state the main theorem in this section.

Theorem 3.2. *Let $(\mathfrak{A}, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum of finite classical type. For $v \in B(\infty)$ (resp. $v \in B(\lambda)$), let $\mathbf{a}(v)$ be the adapted string of v with respect to the expression $w_0 = r_{s_1} \cdots r_{s_n}$ given in Table 1 and let $\nabla(\mathbf{a}(v)) = \nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)$. Then*

- (1) $\text{hd Ind } \nabla(\mathbf{a}(v))$ is irreducible.
- (2) The map $\Psi : B(\infty) \longrightarrow \mathbb{B}(\infty)$ defined by

$$\Psi(v) = \text{hd Ind } \nabla(\mathbf{a}(v)) \quad \text{for } v \in B(\infty)$$

is a crystal isomorphism.

- (3) The map $\Psi^\lambda : B(\lambda) \longrightarrow \mathbb{B}(\lambda)$ defined by

$$\Psi^\lambda(v) = \text{hd Ind } \nabla(\mathbf{a}(v)) \quad \text{for } v \in B(\lambda)$$

is a crystal isomorphism.

Theorem 3.2 and Proposition 3.1 combine to give the following explicit description of the irreducible graded modules over R and R^λ .

Corollary 3.3. *Let $(\mathfrak{A}, P, \Pi, P^\vee, \Pi^\vee)$ be a Cartan datum of finite classical type. Then*

- (1) the set

$$\mathcal{A} = \{\text{hd Ind } \nabla(\mathbf{a}) \mid \mathbf{a} \in \mathcal{S}\}$$

is the complete list of all irreducible graded R -modules up to isomorphism and grading shift.

(2) For $\lambda \in P^+$, the set

$$\mathcal{A}^\lambda = \{\text{hd Ind} \nabla(\mathbf{a}) \mid \mathbf{a} \in \mathcal{S}^\lambda\}$$

is the complete list of all irreducible graded R^λ -modules up to isomorphism and grading shift.

In the following proposition, we give an upper bound for the number of $\nabla_{(a,b)}$'s that can occur in $\nabla(\mathbf{a}(v))$ for $v \in B(\lambda)$.

Proposition 3.4. *For $v \in B(\lambda)$, let $\eta(v)$ be the number of $\nabla_{(a,b)}$'s in the outer tensor product $\nabla(\mathbf{a}(v)) = \nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)$. Then $\eta(v) \leq n\lambda(h)$, where*

$$h = \begin{cases} h_1 + \cdots + h_n & (\mathbf{A}_n), \\ 2h_1 + \cdots + 2h_{n-1} + h_n & (\mathbf{B}_n), \\ 2(h_1 + \cdots + h_n) & (\mathbf{C}_n), \\ 2h_1 + \cdots + 2h_{n-2} + h_{n-1} + h_n & (\mathbf{D}_n). \end{cases}$$

Proof. Let $\mathbf{a} = \mathbf{a}(v)$ and write $\mathbf{a} = \mathbf{a}_1 * \cdots * \mathbf{a}_n$, where $\mathbf{a}_k = (\mathbf{a}_{k,1}, \mathbf{a}_{k,2}, \dots, \mathbf{a}_{k,l_k})$ is the subsequence of \mathbf{a} defined in (2.3). Since $\mathbf{a}_{k,1} = t_{k,n+1-k}$ if \mathfrak{A} is of type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ and $\mathbf{a}_{1,1} = t_{1,n-1}$, $\mathbf{a}_{2,1} = t_{1,n}$, $\mathbf{a}_{k,1} = t_{k-1,n+1-k}$ when $n \geq 3$ for type \mathbf{D}_n , it follows from (3.5) and (3.6) that

$$\eta(v) \leq \mathbf{a}_{1,1} + \cdots + \mathbf{a}_{n-1,1} + \mathbf{a}_{n,1}.$$

Thus, it suffices to show that

$$\mathbf{a}_{1,1} + \cdots + \mathbf{a}_{n-1,1} + \mathbf{a}_{n,1} \leq n\lambda(h).$$

Let $\lambda = \lambda_1 \Lambda_1 + \cdots + \lambda_n \Lambda_n$. If $n > 1$ for type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ and $n > 2$ for type \mathbf{D}_n , then using the description in Proposition 3.1, we will obtain

$$(3.7) \quad \mathbf{a}_{n,1} \leq \lambda(h)$$

from the following inequalities:

$$\begin{aligned}
(\mathbf{A}_n) : \mathbf{a}_{n,1} = t_{n,1} &\leq \lambda_1 + t_{n,2} \leq \lambda_1 + \lambda_2 + t_{n,3} \leq \cdots \leq \lambda_1 + \cdots + \lambda_n = \lambda(h), \\
(\mathbf{B}_n) : \mathbf{a}_{n,1} = t_{n,1} &\leq \lambda_1 + t_{n,2} + t_{n,2n-2} - 2t_{n,2n-1} \leq \lambda_1 + \lambda_2 + t_{n,3} + t_{n,2n-3} - t_{n,2n-2} - t_{n,2n-1} \\
&\leq \cdots \leq \lambda_1 + \cdots + \lambda_{n-1} + t_{n,n} - t_{n,n+1} - t_{n,2n-1} \\
&\leq \lambda_1 + \cdots + \lambda_n + t_{n,n+1} - t_{n,2n-1} \leq \lambda_1 + \cdots + \lambda_n + \lambda_{n-1} + t_{n,n+2} - t_{n,2n-1} \\
&\leq \cdots \leq 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + \lambda_n = \lambda(h), \\
(\mathbf{C}_n) : \mathbf{a}_{n,1} = t_{n,1} &\leq \lambda_1 + t_{n,2} + t_{n,2n-2} - 2t_{n,2n-1} \leq \lambda_1 + \lambda_2 + t_{n,3} + t_{n,2n-3} - t_{n,2n-2} - t_{n,2n-1} \\
&\leq \cdots \leq \lambda_1 + \cdots + \lambda_{n-1} + 2t_{n,n} - t_{n,n+1} - t_{n,2n-1} \\
&\leq \lambda_1 + \cdots + \lambda_{n-1} + 2\lambda_n + t_{n,n+1} - t_{n,2n-1} \\
&\leq \cdots \leq 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-1} + 2\lambda_n = \lambda(h), \\
(\mathbf{D}_n) : \mathbf{a}_{n,1} = t_{n-1,1} &\leq \lambda_1 + t_{n-1,2} + t_{n-1,2n-3} - 2t_{n-1,2n-2} \\
&\leq \lambda_1 + \lambda_2 + t_{n-1,3} + t_{n-1,2n-4} - t_{n-1,2n-3} - t_{n-1,2n-2} \\
&\leq \cdots \leq \lambda_1 + \cdots + \lambda_{n-2} + t_{n-1,n-1} + t_{n-1,n} - t_{n-1,n+1} - t_{n-1,2n-2} \\
&\leq \lambda_1 + \cdots + \lambda_n + t_{n-1,n+1} - t_{n-1,2n-2} \\
&\leq \lambda_1 + \cdots + \lambda_n + \lambda_{n-2} + t_{n-1,n+2} - t_{n-1,2n-2} \\
&\leq \cdots \leq 2\lambda_1 + 2\lambda_2 + \cdots + 2\lambda_{n-2} + \lambda_{n-1} + \lambda_n = \lambda(h).
\end{aligned}$$

We proceed by induction on n . If $n = 2$ for type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ or $n = 3$ for type \mathbf{D}_n , then the assertion can be proved in the same manner as above. We assume that $n > 2$ for type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$ and $n > 3$ for type \mathbf{D}_n . Let $U_q(\mathfrak{g}_{n-1})$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i ($i \in I \setminus \{1\}$) and q^h ($h \in \mathbf{P}^\vee$), and let \mathfrak{B} be the crystal obtained from $B(\lambda)$ by forgetting the 1-arrows. Note that $U_q(\mathfrak{g}_{n-1})$ is of type \mathbf{X}_{n-1} when $U_q(\mathfrak{g})$ is of type \mathbf{X}_n ($\mathbf{X} = \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$). Let $u = \tilde{f}_{\mathbf{s}_n}^{a_n} b_\lambda$ where \mathbf{s}_n is the sequence given in Table 1, and let \mathcal{C} be the connected component of \mathfrak{B} containing v . Then u is the highest weight vector of the $U_q(\mathfrak{g}_{n-1})$ -crystal \mathcal{C} with weight

$$\text{wt}(u) = \begin{cases} \lambda - t_{n,1}\alpha_1 - \cdots - t_{n,n}\alpha_n & (\mathbf{A}_n), \\ \lambda - (t_{n,1} + t_{n,2n-1})\alpha_1 - \cdots - (t_{n,n-1} + t_{n,n+1})\alpha_{n-1} - (t_{n,n})\alpha_n & (\mathbf{B}_n, \mathbf{C}_n), \\ \lambda - (t_{n-1,1} + t_{n-1,2n-2})\alpha_1 - \cdots - (t_{n-1,n-2} + t_{n-1,n+1})\alpha_{n-2} \\ \quad - t_{n-1,n}\alpha_{n-1} - t_{n-1,n-1}\alpha_n & (\mathbf{D}_n). \end{cases}$$

Now suppose that $h' = h - h_1$ (\mathbf{A}_n), and $h' = h - 2h_1$ ($\mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n$). Then we see that

$$\begin{aligned}
\alpha_1(h') &= -1, \quad \alpha_2(h') = 1, \quad \alpha_i(h') = 0 \quad (i = 3, \dots, n) & (\mathbf{A}_n), \\
\alpha_1(h') &= -2, \quad \alpha_2(h') = 2, \quad \alpha_i(h') = 0 \quad (i = 3, \dots, n) & (\mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n).
\end{aligned}$$

Proposition 3.1 implies

$$(3.8) \quad \text{wt}(u)(h') \leq \lambda(h)$$

by the following calculations:

$$\begin{aligned} (\mathbf{A}_n) : \quad & \text{wt}(u)(h') = \lambda(h') + t_{n,1} - t_{n,2} \leq \lambda(h), \\ (\mathbf{B}_n, \mathbf{C}_n) : \quad & \text{wt}(u)(h') = \lambda(h') + 2(t_{n,1} + t_{n,2n-1}) - 2(t_{n,2} + t_{n,2n-2}) \\ & = \lambda(h') + 2(t_{n,1} - t_{n,2} - t_{n,2n-2} + 2t_{n,2n-1}) - 2t_{n,2n-1} \leq \lambda(h), \\ (\mathbf{D}_n) : \quad & \text{wt}(u)(h') = \lambda(h') + 2(t_{n,1} + t_{n,2n-2}) - 2(t_{n,2} + t_{n,2n-3}) \\ & = \lambda(h') + 2(t_{n,1} - t_{n,2} - t_{n,2n-3} + 2t_{n,2n-2}) - 2t_{n,2n-2} \leq \lambda(h). \end{aligned}$$

Since the adapted string of u is $\mathbf{a}_1 * \cdots * \mathbf{a}_{n-1}$, we obtain by the induction hypothesis that

$$\mathbf{a}_{1,1} + \cdots + \mathbf{a}_{n-1,1} \leq (n-1)\text{wt}(u)(h'),$$

which yields, by (3.7) and (3.8),

$$\mathbf{a}_{1,1} + \cdots + \mathbf{a}_{n-1,1} + \mathbf{a}_{n,1} \leq (n-1)\lambda(h) + \lambda(h) = n\lambda(h).$$

□

Example 3.5. Assume $(\mathfrak{A}, \mathbf{P}, \Pi, \mathbf{P}^\vee, \Pi^\vee)$ is type \mathbf{B}_3 and $\lambda = \Lambda_1 + \Lambda_2 + 3\Lambda_3 \in \mathbf{P}^+$. In this case, $w_0 = r_3(r_2r_3r_2)(r_1r_2r_3r_2r_1)$ and

$$\mathbf{B} = \boxed{\overline{1}} \xleftarrow{1} \boxed{\overline{2}} \xleftarrow{2} \boxed{\overline{3}} \xleftarrow{3} \boxed{0} \xleftarrow{3} \boxed{3} \xleftarrow{2} \boxed{2} \xleftarrow{1} \boxed{1}.$$

Kashiwara and Nakashima [17] constructed combinatorial realizations of highest weight crystals for finite classical types using certain semistandard tableaux, which are now referred to as *Kashiwara-Nakashima tableaux*. We take this combinatorial model as a realization of the crystal $B(\lambda)$. Let T denote the following element in $B(\lambda)$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & \overline{3} & \overline{1} \\ \hline \overline{3} & 0 & \overline{1} & \\ \hline 2 & \overline{3} & & \\ \hline \end{array}.$$

Since $w_0 = r_3(r_2r_3r_2)(r_1r_2r_3r_2r_1)$, using the crystal structure given in [17], we have

$$\mathbf{a}(T) = (3, 3, 3, 0, 4, 3, 5, 2, 1) \in \mathcal{S}^\lambda, \quad \Delta(\mathbf{a}(T)) = \begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 5 & 2 & 1 \\ \hline & 3 & 3 & 0 & \\ \hline & & 3 & & \\ \hline \end{array},$$

$$T_5 := \tilde{e}_1^4(T_4) \quad T_6 := \tilde{e}_2^3(T_5) \quad T_7 := \tilde{e}_3^5(T_6) \quad T_8 := \tilde{e}_2^2(T_7) \quad T_9 := \tilde{e}_1^1(T_8)$$

1	1	1	$\overline{2}$
2	3	$\overline{2}$	
3	0		

1	1	1	$\overline{3}$
2	2	$\overline{3}$	
3	0		

1	1	1	3
2	2	3	
3	3		

1	1	1	2
2	2	2	
3	3		

1	1	1	1
2	2	2	
3	3		

$$T_2 := \tilde{e}_2^3(T_1)$$

$$T_3 := \tilde{e}_3^3(T_2)$$

$$T_4 := \tilde{e}_2^0(T_3)$$

1	2	2	$\overline{1}$
2	0	$\overline{1}$	
$\overline{3}$	$\overline{3}$		

1	2	2	$\overline{1}$
2	3	$\overline{1}$	
3	0		

1	2	2	$\overline{1}$
2	3	$\overline{1}$	
3	0		

$$T_1 := \tilde{e}_3^3(T)$$

1	3	3	$\overline{1}$
3	0	$\overline{1}$	
$\overline{2}$	$\overline{3}$		

Note that $\varepsilon_i(T_9) = 0$ ($i = 1, 2, 3$), $\varepsilon_i(T_4) = 0$ ($i = 2, 3$) and $\varepsilon_i(T_1) = 0$ ($i = 3$). It follows from (3.5) and (3.6) that

$$\vartheta_{1,3} = 1, \quad \vartheta_{1,4} = 1,$$

$$\vartheta_{2,2} = 1, \quad \vartheta_{2,3} = 1, \quad \vartheta_{2,4} = 1, \quad \vartheta_{2,5} = 0,$$

$$\vartheta_{3,1} = 1, \quad \vartheta_{3,2} = 0, \quad \vartheta_{3,3} = 1, \quad \vartheta_{3,4} = 0, \quad \vartheta_{3,5} = 1, \quad \vartheta_{3,6} = 1,$$

and using the definition of \hat{i} in (3.2), we have

$$\begin{aligned} \nabla(\mathbf{a}(T); 1) &= \nabla_{(\widehat{3}, \widehat{4})}^{\boxtimes \vartheta_{1,3}} \boxtimes \nabla_{(\widehat{3}, \widehat{5})}^{\boxtimes \vartheta_{1,4}} \\ &= \nabla_{(\overline{3}, 0)} \boxtimes \nabla_{(\overline{3}, 3)}, \\ \nabla(\mathbf{a}(T); 2) &= \nabla_{(\widehat{2}, \widehat{3})}^{\boxtimes \vartheta_{2,2}} \boxtimes \nabla_{(\widehat{2}, \widehat{4})}^{\boxtimes \vartheta_{2,3}} \boxtimes \nabla_{(\widehat{2}, \widehat{5})}^{\boxtimes \vartheta_{2,4}} \boxtimes \nabla_{(\widehat{2}, \widehat{6})}^{\boxtimes \vartheta_{2,5}} \\ &= \nabla_{(\overline{2}, \overline{3})} \boxtimes \nabla_{(\overline{2}, 0)} \boxtimes \nabla_{(\overline{2}, 3)}, \\ \nabla(\mathbf{a}(T); 3) &= \nabla_{(\widehat{1}, \widehat{2})}^{\boxtimes \vartheta_{3,1}} \boxtimes \nabla_{(\widehat{1}, \widehat{3})}^{\boxtimes \vartheta_{3,2}} \boxtimes \nabla_{(\widehat{1}, \widehat{4})}^{\boxtimes \vartheta_{3,3}} \boxtimes \nabla_{(\widehat{1}, \widehat{5})}^{\boxtimes \vartheta_{3,4}} \boxtimes \nabla_{(\widehat{1}, \widehat{6})}^{\boxtimes \vartheta_{3,5}} \boxtimes \nabla_{(\widehat{1}, \widehat{7})}^{\boxtimes \vartheta_{3,6}} \\ &= \nabla_{(\overline{1}, \overline{2})} \boxtimes \nabla_{(\overline{1}, 0)} \boxtimes \nabla_{(\overline{1}, 2)} \boxtimes \nabla_{(\overline{1}, 1)}. \end{aligned}$$

By Theorem 3.2, the R^λ -module M corresponding to T is given as follows:

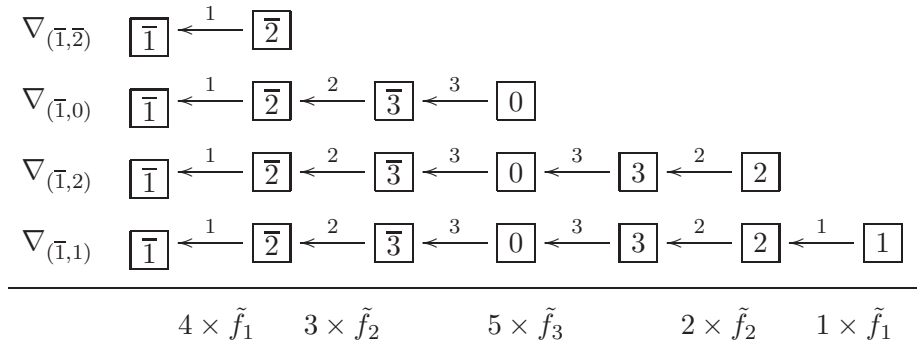
$$\begin{aligned} M &= \Psi^\lambda(T) \\ &= \text{hd Ind}(\nabla(\mathbf{a}(T); 1) \boxtimes \nabla(\mathbf{a}(T); 2) \boxtimes \nabla(\mathbf{a}(T); 3)) \\ &= \text{hd Ind} \left(\nabla_{(\overline{3}, 0)} \boxtimes \nabla_{(\overline{3}, 3)} \boxtimes \nabla_{(\overline{2}, \overline{3})} \boxtimes \nabla_{(\overline{2}, 0)} \boxtimes \nabla_{(\overline{2}, 3)} \boxtimes \nabla_{(\overline{1}, \overline{2})} \boxtimes \nabla_{(\overline{1}, 0)} \boxtimes \nabla_{(\overline{1}, 2)} \boxtimes \nabla_{(\overline{1}, 1)} \right). \end{aligned}$$

In this example, $\eta(T) = 9 < n\lambda(h) = 3(\Lambda_1 + \Lambda_2 + 3\Lambda_3)(2h_1 + 2h_2 + h_3) = 21$.

Recall the definition of $\mathcal{N}_k(T)$ in (1.4) (here the element v in $B(\lambda)$ is the tableau T). Then

$$\mathcal{N}_1(T) = \text{hd Ind}(\nabla(\mathbf{a}(T); 1)), \quad \mathcal{N}_2(T) = \text{hd Ind}(\nabla(\mathbf{a}(T); 2)), \quad \mathcal{N}_3(T) = \text{hd Ind}(\nabla(\mathbf{a}(T); 3)).$$

We will prove that such a realization of these modules exists for all finite classical types in the next section (Lemma 4.3). In this example, we give an intuitive picture for the case of $\mathcal{N}_3(T)$. For $a, b \in \mathbf{B}$ with $a \succ b$, by the definition of $\nabla_{(a,b)}$ in (3.3) we can identify $\nabla_{(a,b)}$ with the segment of \mathbf{B} between a and b . Then we have the following diagram.



The Kashiwara operators at the bottom of the diagram are obtained by adding up vertically the number of i -colored arrows in the segments. By Proposition 2.3, we know

$$\mathcal{N}_3(T) = \tilde{f}_1^4 \tilde{f}_2^3 \tilde{f}_3^5 \tilde{f}_2^2 \tilde{f}_1^1 \mathbf{1}.$$

Since $\mathcal{N}_3(T) = \text{hd Ind}(\nabla_{(\bar{1}, \bar{2})} \boxtimes \nabla_{(\bar{1}, 0)} \boxtimes \nabla_{(\bar{1}, 2)} \boxtimes \nabla_{(\bar{1}, 1)})$, taking the head of $\text{Ind}(\nabla_{(\bar{1}, \bar{2})} \boxtimes \nabla_{(\bar{1}, 0)} \boxtimes \nabla_{(\bar{1}, 2)} \boxtimes \nabla_{(\bar{1}, 1)})$ can be understood as summing vertically the i -colored arrows in the segments corresponding to the modules $\nabla_{(\bar{1}, \bar{2})}$, $\nabla_{(\bar{1}, 0)}$, $\nabla_{(\bar{1}, 2)}$ and $\nabla_{(\bar{1}, 1)}$.

Now from the crystal structure in [17], we know that

$$\tilde{f}_1 T = \begin{array}{|c|c|c|c|} \hline 2 & 3 & \bar{3} & \bar{1} \\ \hline \bar{3} & 0 & \bar{1} & \\ \hline \bar{1} & \bar{3} & & \\ \hline \end{array}, \quad \tilde{f}_2 T = \begin{array}{|c|c|c|c|} \hline 1 & 3 & \bar{3} & \bar{1} \\ \hline \bar{3} & 0 & \bar{1} & \\ \hline \bar{2} & \bar{2} & & \\ \hline \end{array}, \quad \tilde{f}_3 T = 0,$$

which yield $\mathbf{a}(\tilde{f}_1 T) = (3, 2, 1, 0, 5, 4, 7, 2, 1) \in \mathcal{S}^\lambda$, $\mathbf{a}(\tilde{f}_2 T) = (3, 4, 3, 0, 4, 3, 5, 2, 1) \in \mathcal{S}^\lambda$ and

$$\Delta(\mathbf{a}(\tilde{f}_1 T)) = \begin{array}{|c|c|c|c|c|} \hline 5 & 4 & 7 & 2 & 1 \\ \hline & 2 & 1 & 0 & \\ \hline & & 3 & & \\ \hline \end{array}, \quad \Delta(\mathbf{a}(\tilde{f}_2 T)) = \begin{array}{|c|c|c|c|c|} \hline 4 & 3 & 5 & 2 & 1 \\ \hline & 4 & 3 & 0 & \\ \hline & & 3 & & \\ \hline \end{array}.$$

Therefore, by Theorem 3.2, the irreducible modules $\tilde{f}_1(M) = \text{hd lnd}(L(1) \boxtimes M)$ and $\tilde{f}_2(M) = \text{hd lnd}(L(2) \boxtimes M)$ can be gotten as follows:

$$\begin{aligned} \tilde{f}_1(M) &= \tilde{f}_1(\Psi^\lambda(T)) = \Psi^\lambda(\tilde{f}_1 T) \\ &= \text{hd lnd} \left(\nabla_{(\bar{3},0)} \boxtimes \nabla_{(\bar{3},3)} \boxtimes \nabla_{(\bar{2},\bar{3})} \boxtimes \nabla_{(\bar{2},0)} \boxtimes \nabla_{(\bar{1},\bar{2})} \boxtimes \nabla_{(\bar{1},0)} \boxtimes \nabla_{(\bar{1},3)} \boxtimes \nabla_{(\bar{1},2)} \boxtimes \nabla_{(\bar{1},1)} \right), \\ \tilde{f}_2(M) &= \tilde{f}_2(\Psi^\lambda(T)) = \Psi^\lambda(\tilde{f}_2 T) \\ &= \text{hd lnd} \left(\nabla_{(\bar{3},0)} \boxtimes \nabla_{(\bar{3},3)} \boxtimes \nabla_{(\bar{2},\bar{3})}^{\boxtimes 2} \boxtimes \nabla_{(\bar{2},0)} \boxtimes \nabla_{(\bar{2},3)} \boxtimes \nabla_{(\bar{1},\bar{2})} \boxtimes \nabla_{(\bar{1},0)} \boxtimes \nabla_{(\bar{1},2)} \boxtimes \nabla_{(\bar{1},1)} \right). \end{aligned}$$

Note that by the same argument as above, one can compute the action of Kashiwara operators on $\mathbb{B}(\infty)$ explicitly using the combinatorial realizations of $B(\infty)$ for finite classical type in [4, 8].

4. PROOF OF THEOREM 3.2

This section is devoted to a proof of Theorem 3.2. We first give a sufficient condition for $\text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(c,d)}) \simeq \text{Ind}(\nabla_{(c,d)} \boxtimes \nabla_{(a,b)})$ to hold (Lemma 4.1). Then we extend the result in [14, Lem. 4.3] for type A_n to the other finite classical types in Lemma 4.2 below, and prove Lemma 4.3 which yields $\mathcal{N}_n(v) = \text{hd lnd} \nabla(\mathbf{a}(v); n)$ for $v \in B(\infty)$. From that we can deduce $\mathcal{N}_k(v) = \text{hd lnd} \nabla(\mathbf{a}(v); k)$ for all k . Using Lemma 1.8, Proposition 1.10, and Lemma 4.3, we argue that the map $\Psi : B(\infty) \rightarrow \mathbb{B}(\infty)$ is a crystal isomorphism. It then follows from the crystal embedding $\iota_\lambda : B(\lambda) \rightarrow B(\infty) \otimes T_\lambda \otimes C$ in (1.1) that the map $\Psi^\lambda : B(\lambda) \rightarrow \mathbb{B}(\lambda)$ is a crystal isomorphism too.

Given $\mathbf{i} \in I^\alpha$ and $\mathbf{j} \in I^\beta$, a sequence $\mathbf{k} \in I^{\alpha+\beta}$ is called a *shuffle* of \mathbf{i} and \mathbf{j} if \mathbf{k} is a permutation of $\mathbf{i} * \mathbf{j}$ such that \mathbf{i} and \mathbf{j} are subsequences of \mathbf{k} . For $X = \sum x_{\mathbf{i}} \mathbf{i}$ and $Y = \sum y_{\mathbf{j}} \mathbf{j}$, we define $X \star Y$ by

$$X \star Y = \sum_{\mathbf{k}} x_{\mathbf{i}} y_{\mathbf{j}} \mathbf{k},$$

where \mathbf{k} runs over all the shuffles of \mathbf{i} and \mathbf{j} . Suppose $M \in R(\alpha)\text{-fmod}$ and $N \in R(\beta)\text{-fmod}$, and assume Q is a quotient (resp. L is a submodule) of $\text{Ind}(M \boxtimes N)$. Then

$$(4.1) \quad \text{ch}(\text{Ind}(M \boxtimes N)) = \text{ch}(M) \star \text{ch}(N),$$

$$(4.2) \quad \text{any term of } \text{ch} Q \text{ (resp. } \text{ch} L \text{) is a shuffle of some } \mathbf{i} \in \text{ch}(M) \text{ and } \mathbf{j} \in \text{ch}(N).$$

The quantum Serre relations (1.3) imply that

$$(4.3) \quad \dim(e(\dots, i, j, \dots)M) = \dim(e(\dots, j, i, \dots)M) \quad \text{if } a_{ij} = 0,$$

$$(4.4) \quad \begin{aligned} 2\dim(e(\dots, i, j, i, \dots)M) \\ = \dim(e(\dots, i, i, j, \dots)M) + \dim(e(\dots, j, i, i, \dots)M) \end{aligned} \quad \text{if } a_{ij} = -1,$$

$$(4.5) \quad \begin{aligned} \dim(e(\dots, i, i, i, j, \dots)M) + 3\dim(e(\dots, i, j, i, i, \dots)M) \\ = \dim(e(\dots, j, i, i, i, \dots)M) + 3\dim(e(\dots, i, i, j, i, \dots)M) \end{aligned} \quad \text{if } a_{ij} = -2,$$

for $M \in R\text{-fmod}$.

Using the description of $\nabla_{(a,b)}$ in Section 3, we obtain a surjective homomorphism

$$(4.6) \quad \text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(b,c)}) \twoheadrightarrow \nabla_{(a,c)}$$

for $a, b, c \in \mathbf{B}$ with $a \succ b \succ c$.

Lemma 4.1. *Let $a, b, c, d \in \mathbf{B}$ with $a \succeq c \succ d \succeq b$. Suppose that one of the following conditions hold:*

- (1) \mathfrak{A} is of type A_n ,
- (2) \mathfrak{A} is of type B_n and one of the following three holds:
 - (i) $0 \succeq a$ and $c \neq 0$,
 - (ii) $b \succeq 0$ and $d \neq 0$,
 - (iii) $a \succeq \bar{n}$, $n \succeq b$ and either $c = 0$ or $d = 0$,
- (3) \mathfrak{A} is of type C_n and one of the following three holds:
 - (i) $b \succeq \bar{n}$,
 - (ii) $a = c = \bar{n}$,
 - (iii) $a = \bar{1}$ and $c = \bar{n}$,
- (4) \mathfrak{A} is of type D_n and one of the following three holds:
 - (i) $b \succeq \overline{n-1}$,
 - (ii) $a = c = \overline{n-1}$,
 - (iii) $a = \bar{1}$ and $c = \overline{n-1}$.

Then

$$\text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(c,d)}) \simeq \text{Ind}(\nabla_{(c,d)} \boxtimes \nabla_{(a,b)})$$

and these modules are irreducible.

Proof. Let

$$M = \text{Ind}(\nabla_{(a,b)} \boxtimes \nabla_{(c,d)}), \quad N = \text{Ind}(\nabla_{(c,d)} \boxtimes \nabla_{(a,b)}),$$

and let Q (resp. L) be any nonzero quotient (resp. submodule) of M . We will assume that \mathbf{k} is a certain sequence of elements in I such that $\mathbf{k} \in \text{ch}M$, $\mathbf{k} \in \text{ch}Q$, and $\mathbf{k} \in \text{ch}L$. Let

ξ_M (resp. ξ_Q , ξ_L) be the multiplicity of \mathbf{k} in $\text{ch}M$ (resp. $\text{ch}Q$, $\text{ch}L$). If we can show that $\xi_M = \xi_Q = \xi_L$, we can conclude that M is irreducible and $M \simeq N$, since $\text{ch}M = \text{ch}N$.

For type A_n , the assertion follows from [14, Lem. 4.1]. We assume that \mathfrak{A} is of type B_n . If either (i) or (ii) holds, the proof is identical to that for type A_n . So suppose that (iii) holds. Without loss of generality, we may assume that $d = 0$. Write $\mathbf{i} = \mathbf{i}(a, b) = (i, \dots, n, n, \dots, j)$, $\mathbf{j} = \mathbf{i}(c, d) = (p, p+1, \dots, n-1, n)$, and let

$$\mathbf{k} = (i, \dots, p, p, p+1, p+1, \dots, n-1, n-1, n, n, n, n-1, n-2, \dots, j).$$

Note that \mathbf{k} is a shuffle of \mathbf{i} and \mathbf{j} . Let l be the length of \mathbf{j} . It follows from (3.4) and (4.1) that the term \mathbf{k} appears in $\text{ch}M$ with multiplicity $\xi_M = 2 \times 2^{l-1}3$.

By Lemma 1.5 and (3.4), $\mathbf{i} * \mathbf{j}$ appears in the character $\text{ch}Q$ of any quotient module $Q \subseteq M$. We claim that \mathbf{k} also appears in $\text{ch}Q$. Note that $a_{n,n-1} = -2$ for type B_n .

We first assume that $p \neq n-1, n$. By the quantum Serre relations (1.3), if $j > p+1$, then

$$\begin{aligned} \mathbf{i} * \mathbf{j} &= (i, \dots, n, n, \dots, j, p, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.3)}{\Rightarrow} (i, \dots, p, p+1, p, \dots, n, n, \dots, j, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, p+1, \dots, n, n, \dots, j, p+1, \dots, n) \in \text{ch}Q. \end{aligned}$$

If $j = p+1$, then

$$\begin{aligned} \mathbf{i} * \mathbf{j} &= (i, \dots, n, n, \dots, p+1, p, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, n, n, \dots, p, p+1, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.3)}{\Rightarrow} (i, \dots, p, p+1, p, \dots, n, n, \dots, p+1, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, p+1, \dots, n, n, \dots, p+1, p+1, \dots, n) \in \text{ch}Q. \end{aligned}$$

If $j = p$, then

$$\begin{aligned} \mathbf{i} * \mathbf{j} &= (i, \dots, n, n, \dots, p+1, p, p, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.4)}{\Rightarrow} (i, \dots, n, n, \dots, p, p+1, p, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.3)}{\Rightarrow} (i, \dots, p, p+1, p, \dots, n, n, \dots, p, p+1, \dots, n) \in \text{ch}Q, \\ &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, p+1, \dots, n, n, \dots, p, p+1, \dots, n) \in \text{ch}Q. \end{aligned}$$

If $j < p$, then

$$\begin{aligned}
 \mathbf{i} * \mathbf{j} &= (i, \dots, n, n, \dots, j, p, p+1, \dots, n) \in \text{ch}Q, \\
 &\stackrel{(4.4)}{\Rightarrow} (i, \dots, n, n, \dots, p+1, p, p-1, p, \dots, j, p+1, \dots, n) \in \text{ch}Q, \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, n, n, \dots, p+1, p, p, p-1, \dots, j, p+1, \dots, n) \in \text{ch}Q, \\
 &\stackrel{(4.4)}{\Rightarrow} (i, \dots, n, n, \dots, p, p+1, p, p-1, \dots, j, p+1, \dots, n) \in \text{ch}Q, \\
 &\stackrel{(4.3)}{\Rightarrow} (i, \dots, p, p+1, p, \dots, n, n, \dots, j, p+1, \dots, n) \in \text{ch}Q, \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, p+1, \dots, n, n, \dots, j, p+1, \dots, n) \in \text{ch}Q.
 \end{aligned}$$

Applying this argument repeatedly, we determine that

$$\begin{aligned}
 &(i, \dots, p, p, \dots, n-2, n-2, n-1, n, n, \dots, j, n-1, n) \in \text{ch}Q \\
 &\stackrel{(4.2), (4.3), (4.4)}{\Rightarrow} (i, \dots, p, p, \dots, n-1, n, n, n-1, n-1, \dots, j, n) \in \text{ch}Q, \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, \dots, n-1, n, n-1, n, n-1, \dots, j, n) \in \text{ch}Q \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, \dots, p, p, \dots, n-1, n-1, n, n, n-1, \dots, j, n) \in \text{ch}Q, \\
 &\stackrel{(4.3)}{\Rightarrow} (i, \dots, p, p, \dots, n-1, n-1, n, n, n-1, n, \dots, j) \in \text{ch}Q, \\
 &\stackrel{(4.2), (4.5)}{\Rightarrow} \mathbf{k} = (i, \dots, p, p, \dots, n-1, n-1, n, n, n, n-1, \dots, j) \in \text{ch}Q.
 \end{aligned}$$

We suppose that $p = n-1, n$. Then, the same argument as above gives

$$\mathbf{k} = (i, \dots, n-1, n-1, n, n, n, n-1, \dots, j) \in \text{ch}Q.$$

Now for any nonzero submodule L of M , $\text{ch}L$ contains $\mathbf{j} * \mathbf{i}$ by Lemma 1.5 and (3.4). If $a = c$, then

$$\begin{aligned}
 \mathbf{j} * \mathbf{i} &= (i, i+1, \dots, n, i, \dots, n, n, \dots, j) \in \text{ch}L, \\
 &\stackrel{(4.3)}{\Rightarrow} (i, i+1, i, \dots, n, i+1, \dots, n, n, \dots, j) \in \text{ch}L, \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, i, i+1, \dots, n, i+1, \dots, n, n, \dots, j) \in \text{ch}L, \\
 &\quad \vdots \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} (i, i, i+1, i+1, \dots, n-2, n-2, n-1, n, n-1, n, n, \dots, j) \in \text{ch}L, \\
 &\stackrel{(4.2), (4.4)}{\Rightarrow} \mathbf{k} = (i, i, i+1, i+1, \dots, n-2, n-2, n-1, n-1, n, n, n, \dots, j) \in \text{ch}L.
 \end{aligned}$$

If $a \succ c$, then

$$\begin{aligned}
\mathbf{j} * \mathbf{i} &= (p, \dots, n-1, n, i, \dots, n, n, \dots, j) \in \text{ch}L, \\
&\stackrel{(4.3)}{\Rightarrow} (p, \dots, n-1, i, \dots, n, n-1, n, n, \dots, j) \in \text{ch}L, \\
&\stackrel{(4.2), (4.5)}{\Rightarrow} (p, \dots, n-1, i, \dots, n-1, n, n, n, \dots, j) \in \text{ch}L, \\
&\quad \vdots \\
&\stackrel{(4.2), (4.4)}{\Rightarrow} \mathbf{k} = (i, \dots, p, p, p+1, p+1, \dots, n-1, n-1, n, n, n, \dots, j) \in \text{ch}L.
\end{aligned}$$

Hence, $\text{ch}L$ contains \mathbf{k} in either event. From the structure of \mathbf{k} we see that it occurs in $\text{ch}Q$ (resp. $\text{ch}L$) with multiplicity $\xi_Q \geq 2^{l-1}3!$ (resp. $\xi_L \geq 2^{l-1}3!$). Therefore, since $\xi_M = 2 \times 2^{l-1}3$, we have $\xi_M = \xi_Q = \xi_L$, which implies the result for type B_n .

Now assume that \mathfrak{A} is of type C_n . If either (i) or (ii) holds, the proof is identical to that for type A_n . We assume that (iii) holds. Write $\mathbf{i} = \mathbf{i}(a, b) = (1, 2, \dots, n, \dots, j)$, $\mathbf{j} = \mathbf{i}(c, d) = (n, n-1, \dots, q)$. Then

$$\mathbf{k} := (1, 2, \dots, n-1, n, n, n-1, n-1, \dots, q, q, \dots, j).$$

is a shuffle of \mathbf{i} and \mathbf{j} . Let l be the length of \mathbf{j} . Then (3.4) and (4.1) imply that the term \mathbf{k} appears in $\text{ch}M$ with multiplicity $\xi_M = 2^l$.

For any nonzero quotient Q of M , $\text{ch}Q$ contains $\mathbf{i} * \mathbf{j}$ by (3.4) and Lemma 1.5. We claim that \mathbf{k} occurs in $\text{ch}Q$ as a term. Note that $a_{n-1, n} = -2$. The quantum Serre relations (1.3) and (4.2) imply

$$\begin{aligned}
\mathbf{i} * \mathbf{j} &= (1, \dots, n, \dots, j, n, \dots, q) \in \text{ch}Q, \\
&\stackrel{(4.3)}{\Rightarrow} (1, \dots, n-1, n, n-1, n, \dots, j, n-1, \dots, q) \in \text{ch}Q, \\
&\stackrel{(4.2), (4.4)}{\Rightarrow} (1, \dots, n-1, n, n, n-1, \dots, j, n-1, \dots, q) \in \text{ch}Q.
\end{aligned}$$

Continuing this reasoning gives

$$\mathbf{k} = (1, \dots, n, n, \dots, q, q, \dots, j) \in \text{ch}Q.$$

For any nonzero submodule L of M , $\text{ch}L$ contains $\mathbf{j} * \mathbf{i}$ by Lemma 1.5 and (3.4). Then, using a similar argument, we have

$$\begin{aligned}
\mathbf{j} * \mathbf{i} &= (n, \dots, q, 1, 2, \dots, n, \dots, j) \in \text{ch}L, \\
&\stackrel{(4.2), (4.3), (4.4)}{\Rightarrow} (n, n-1, 1, \dots, n-1, n, n-1, \dots, q, q, \dots, j) \in \text{ch}L, \\
&\stackrel{(4.2), (4.3), (4.4)}{\Rightarrow} (n, 1, \dots, n-2, n-1, n-1, n, n-1, \dots, q, q, \dots, j) \in \text{ch}L,
\end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4.2),(4.5)}{\implies} (n, 1, \dots, n-2, n-1, n, n-1, n-1, \dots, q, q, \dots, j) \in \text{ch}L, \\
 & \stackrel{(4.3)}{\implies} (1, \dots, n, n-1, n, n-1, n-1, \dots, q, q, \dots, j) \in \text{ch}L, \\
 & \stackrel{(4.2),(4.4)}{\implies} \mathbf{k} = (1, \dots, n, n, \dots, q, q, \dots, j) \in \text{ch}L.
 \end{aligned}$$

As \mathbf{k} occurs in $\text{ch}Q$ (resp. $\text{ch}L$) with multiplicity $\xi_Q \geq 2^l$ (resp. $\xi_L \geq 2^l$), and since $\xi_M = 2^l$, the assertion follows from $\xi_M = \xi_Q = \xi_L$.

Lastly, we assume that \mathfrak{A} is of type D_n . If (i) holds, then the proof is identical to that for type A_n .

Suppose that (ii) holds. Write $\mathbf{i} = \mathbf{i}(a, b) = (n, n-1, n-2, \dots, j)$, $\mathbf{j} = \mathbf{i}(c, d) = (n, n-1, n-2, \dots, q)$. Then

$$\mathbf{k} := (n, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j).$$

is a shuffle of \mathbf{i} and \mathbf{j} . Let l be the length of \mathbf{j} . The equations (3.4) and (4.1) imply that the term \mathbf{k} appears in $\text{ch}M$ with multiplicity $\xi_M = 2^l$.

By Lemma 1.5 and (3.4), $\mathbf{i} * \mathbf{j}$ appears in the character $\text{ch}Q$ of any quotient module $Q \subseteq M$. We will show that \mathbf{k} also appears in $\text{ch}Q$. Note that $a_{n,n-2} = a_{n-2,n} = a_{n-1,n-2} = a_{n-2,n-1} = -1$ and $a_{n-1,n} = a_{n,n-1} = 0$. The quantum Serre relations (1.3) imply

$$\begin{aligned}
 \mathbf{i} * \mathbf{j} &= (n, n-1, n-2, \dots, j, n, n-1, n-2, \dots, q) \in \text{ch}Q, \\
 &\stackrel{(4.3)}{\implies} (n-1, n, n-2, n, \dots, j, n-1, n-2, \dots, q) \in \text{ch}Q, \\
 &\stackrel{(4.2),(4.3),(4.4)}{\implies} (n, n, n-1, n-2, \dots, j, n-1, n-2, \dots, q) \in \text{ch}Q, \\
 &\stackrel{(4.2),(4.4)}{\implies} (n, n, n-1, n-2, n-1, \dots, j, n-2, \dots, q) \in \text{ch}Q, \\
 &\stackrel{(4.2),(4.5)}{\implies} (n, n, n-1, n-1, n-2, \dots, j, n-2, \dots, q) \in \text{ch}Q.
 \end{aligned}$$

Continuing this reasoning gives

$$\mathbf{k} = (n, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}Q.$$

For any nonzero submodule L , since $\text{ch}L$ contains $\mathbf{j} * \mathbf{i}$ by Lemma 1.5, the same argument shows that $\text{ch}L$ contains \mathbf{k} . Since \mathbf{k} occurs in $\text{ch}Q$ (resp. $\text{ch}L$) with multiplicity $\xi_Q \geq 2^l$ (resp. $\xi_L \geq 2^l$), the assertion follows from $\xi_M = \xi_Q = \xi_L$.

We now assume that (iii) holds. Since the proof is identical to that for type A_n if $b = \bar{n}$ or $b = n$, we may assume that $n-1 \succeq b$. Write $\mathbf{i} = \mathbf{i}(a, b) = (1, \dots, n-2, n, n-1, n-2, \dots, j)$, $\mathbf{j} = \mathbf{i}(c, d) = (n, n-1, n-2, \dots, q)$ and

$$\mathbf{k} := (1, \dots, n-2, n, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j).$$

Let l be the length of \mathbf{j} . Then \mathbf{k} is a shuffle of \mathbf{i} and \mathbf{j} and appears in $\text{ch}M$ with multiplicity $\xi_M = 2^l$.

For any nonzero quotient Q of M , $\text{ch}Q$ contains $\mathbf{i} * \mathbf{j}$ by (3.4) and Lemma 1.5. Using the same argument as in case (ii), we obtain

$$\mathbf{k} = (1, \dots, n-2, n, n, n-1, n-1, n-2, \dots, j, n-2, \dots, q) \in \text{ch}Q.$$

For any nonzero submodule L of M , $\text{ch}L$ contains $\mathbf{j} * \mathbf{i}$ by Lemma 1.5 and (3.4). By the same argument as in the C_n case, we have

$$\begin{aligned} \mathbf{j} * \mathbf{i} &= (n, \dots, q-1, q, 1, \dots, n-2, n, n-1, n-2, \dots, j) \in \text{ch}L, \\ &\xrightarrow{(4.2), (4.3), (4.4)} (n, n-1, n-2, 1, \dots, n-2, n, n-1, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xrightarrow{(4.2), (4.4)} (n, n-1, 1, \dots, n-2, n-2, n, n-1, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xRightarrow{(4.4)} (n, n-1, 1, \dots, n-2, n, n-2, n-1, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xrightarrow{(4.2), (4.4)} (n, n-1, 1, \dots, n-2, n, n-1, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xRightarrow{(4.3)} (n, 1, \dots, n-1, n-2, n-1, n, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xrightarrow{(4.2), (4.4)} (n, 1, \dots, n-2, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xRightarrow{(4.3)} (1, \dots, n, n-2, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}L, \\ &\xrightarrow{(4.2), (4.4)} \mathbf{k} = (1, \dots, n-2, n, n, n-1, n-1, n-2, n-2, \dots, q, q, \dots, j) \in \text{ch}L. \end{aligned}$$

Since the sequence \mathbf{k} occurs in $\text{ch}Q$ (resp. $\text{ch}L$) with multiplicity $\xi_Q \geq 2^l$ (resp. $\xi_L \geq 2^l$), by $\xi_M = 2^l$, the assertion follows from $\xi_M = \xi_Q = \xi_L$. \square

For $a, b \in \mathbf{B}$, we define

$$(4.7) \quad \delta(a \succeq b) = \begin{cases} 1 & \text{if } a \succeq b, \\ 0 & \text{otherwise.} \end{cases}$$

In the A_n case, it follows from [14, Lem. 4.3] that, for $d_k \in \mathbb{Z}_{\geq 0}$ and $b_k \in \mathbf{B}$ with $b_1 \succ b_2 \succ \dots \succ b_m$,

$$(4.8) \quad \tilde{f}_1^{t_1} \tilde{f}_2^{t_2} \dots \tilde{f}_n^{t_n} \mathbf{1} \simeq \text{Ind} \left(\bigotimes_{k=1}^m \nabla_{(\bar{1}, b_k)}^{\boxtimes d_k} \right),$$

where $t_i = \sum_{j=1}^m d_j \delta(\bar{i} \succ b_j)$ for $i = 1, \dots, n$. Next we will extend this result to the other finite classical types. We first divide the crystal \mathbf{B} for types B_n, C_n, D_n into two pieces so that each part is almost the same as the crystal or the dual of the crystal for type A . For type B_n (resp. C_n, D_n), the crystal \mathbf{B} is cut at the element 0 (resp. $\bar{n}, \overline{n-1}$). In the next result we give an analogue of (4.8) for each part.

Lemma 4.2. *Let $b_k \in \mathbf{B}$, and $d_k \in \mathbb{Z}_{\geq 0}$ for $k = 1, \dots, m$. Set*

$$b_{\bullet} = \begin{cases} \overline{n+1} & (\mathbf{A}_n), \\ 0 & (\mathbf{B}_n), \\ \overline{n} & (\mathbf{C}_n), \\ \overline{n-1} & (\mathbf{D}_n). \end{cases}$$

(1) *If $b_1 \succ b_2 \succ \dots \succ b_m \succeq b_{\bullet}$, then*

$$\tilde{f}_1^{t_1} \tilde{f}_2^{t_2} \dots \tilde{f}_l^{t_l} \mathbf{1} \simeq \text{Ind} \left(\bigotimes_{k=1}^m \nabla_{(\overline{1}, b_k)}^{\boxtimes d_k} \right),$$

where $l = n$ ($\mathbf{A}_n, \mathbf{B}_n$), $l = n-1$ (\mathbf{C}_n), $l = n-2$ (\mathbf{D}_n) and $t_i := \sum_{j=1}^m d_j \delta(\overline{i} \succ b_j)$ for $i = 1, \dots, l$.

(2) *Let*

$$t_i = \begin{cases} \sum_{j=1}^m d_j \delta(i \succeq b_j) & \text{if } i = 1, \dots, n \text{ } (\mathbf{B}_n, \mathbf{C}_n), \quad i = 1, \dots, n-2 \text{ } (\mathbf{D}_n), \\ \sum_{j=1}^m d_j \delta(\overline{n} \succeq b_j) & \text{if } i = n-1 \text{ } (\mathbf{D}_n), \\ \sum_{j=1}^m d_j \delta(n \succeq b_j) & \text{if } i = n \text{ } (\mathbf{D}_n). \end{cases}$$

Then we have

(i) *if $b_{\bullet} \succ b_1 \succ b_2 \succ \dots \succ b_m$ for type \mathbf{B}_n , then*

$$\tilde{f}_n^{s+t_n} \tilde{f}_{n-1}^{t_{n-1}} \dots \tilde{f}_1^{t_1} \mathbf{1} \simeq \text{hd Ind} \left(L(n^s) \boxtimes \left(\bigotimes_{k=1}^m \nabla_{(b_{\bullet}, b_k)}^{\boxtimes d_k} \right) \right) \quad \text{for all } s \in \mathbb{Z}_{\geq 0},$$

and the head occurs with multiplicity one as a composition factor of

$$\text{Ind} \left(L(n^s) \boxtimes \left(\bigotimes_{k=1}^m \nabla_{(b_{\bullet}, b_k)}^{\boxtimes d_k} \right) \right),$$

(ii) *if $b_{\bullet} \succ b_m \succ b_{m-1} \succ \dots \succ b_1$ for type $\mathbf{C}_n, \mathbf{D}_n$, then*

$$\tilde{f}_n^{t_n} \dots \tilde{f}_2^{t_2} \tilde{f}_1^{t_1} \mathbf{1} \simeq \text{Ind} \left(\bigotimes_{k=1}^m \nabla_{(b_{\bullet}, b_k)}^{\boxtimes d_k} \right).$$

Proof. For $k_1, \dots, k_n \in \mathbb{Z}_{\geq 0}$, it follows from Lemma 1.8 that

$$(4.9) \quad \tilde{f}_1^{k_1} \tilde{f}_2^{k_2} \dots \tilde{f}_n^{k_n} \mathbf{1} \simeq \text{hd Ind} \left(L(1^{k_1}) \boxtimes \dots \boxtimes L(n^{k_n}) \right),$$

$$(4.10) \quad \tilde{f}_n^{k_n} \tilde{f}_{n-1}^{k_{n-1}} \dots \tilde{f}_1^{k_1} \mathbf{1} \simeq \text{hd Ind} \left(L(n^{k_n}) \boxtimes \dots \boxtimes L(1^{k_1}) \right).$$

(1) When \mathfrak{A} is of type $\mathbf{A}_n, \mathbf{C}_n, \mathbf{D}_n$, this can be shown in the same way as for [14, Lem. 4.3].

Suppose that \mathfrak{A} is of type \mathbf{B}_n . Note that $\nabla_{(\overline{1}, b_k)}$ is one of the cuspidal representations given in [6, Sec. 6.2] and [20, Sec. 8.5]. By [20, Lem. 6.6], we have $N_k := \text{Ind} \nabla_{(\overline{1}, b_k)}^{\boxtimes d_k}$ is irreducible for $k = 1, \dots, m$. We write $\mathbf{i}(\overline{1}, b_k) = (1, 2, \dots, l_k)$ and let

$$\mathbf{k}_k = (\underbrace{1, \dots, 1}_{d_k}, \underbrace{2, \dots, 2}_{d_k}, \dots, \underbrace{l_k, \dots, l_k}_{d_k})$$

for $k = 1, \dots, m$. Then \mathbf{k}_k appears in $\text{ch}N_k$ with multiplicity $(d_k!)^{l_k}$. Since $b_1 \succ b_2 \succ \dots \succ b_m$, the sequence $\mathbf{k} = \mathbf{k}_m * \dots * \mathbf{k}_1$ appears in $\text{ch}(\text{Ind}(N_m \boxtimes \dots \boxtimes N_1))$ with multiplicity $(d_m!)^{l_m} \dots (d_1!)^{l_1}$. Lemma 1.7 and Lemma 4.1 imply that $\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right) \simeq \text{Ind}(N_m \boxtimes \dots \boxtimes N_1)$ is irreducible.

Now for $\mathbf{t} := (\underbrace{1, \dots, 1}_{t_1}, \underbrace{2, \dots, 2}_{t_2}, \dots, \underbrace{n, \dots, n}_{t_n})$, we have $e(\mathbf{t}) \left(\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right) \right) \neq 0$. Since $\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ is irreducible, it follows from Lemma 1.5 that there is a surjective homomorphism

$$\text{Ind}(L(1^{t_1}) \boxtimes \dots \boxtimes L(n^{t_n})) \twoheadrightarrow \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right).$$

Therefore, $\tilde{f}_1^{t_1} \dots \tilde{f}_n^{t_n} \mathbf{1} \simeq \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ by (4.9).

(2) For this part, note that since $\nabla_{(b_\bullet, b_k)}$ is a cuspidal representation as in [6, Sec. 6] and [20, Sec. 8], $N_k := \text{Ind}\nabla_{(b_\bullet, b_k)}^{\boxtimes d_k}$ is irreducible by [20, Lem. 6.6]. We now separate considerations according to the type.

(Case B_n) When \mathfrak{A} is of type B_n , let $N = \text{Ind}\left(\nabla_{(n, b_m)}^{\boxtimes d_m} \boxtimes \dots \boxtimes \nabla_{(n, b_1)}^{\boxtimes d_1}\right)$. By [14, Lem. 4.3], we have $\tilde{f}_{n-1}^{t_{n-1}} \dots \tilde{f}_1^{t_1} \mathbf{1} \simeq N$. By (4.6) and Lemma 4.1, for $d \in \mathbb{Z}_{\geq 0}$ we obtain

$$\begin{aligned} \text{Ind}\left(L(n^d) \boxtimes \nabla_{(n, b_k)}^{\boxtimes d}\right) &\simeq \text{Ind}\left(L(n^{d-1}) \boxtimes L(n) \boxtimes \nabla_{(n, b_k)} \boxtimes \nabla_{(n, b_k)}^{\boxtimes d-1}\right) \\ &\twoheadrightarrow \text{Ind}\left(L(n^{d-1}) \boxtimes \nabla_{(0, b_k)} \boxtimes \nabla_{(n, b_k)}^{\boxtimes d-1}\right) \simeq \text{Ind}\left(L(n^{d-1}) \boxtimes \nabla_{(n, b_k)}^{\boxtimes d-1} \boxtimes \nabla_{(0, b_k)}\right) \\ &\vdots \\ &\twoheadrightarrow \text{Ind}\left(\nabla_{(0, b_k)}^{\boxtimes d}\right). \end{aligned}$$

Since $t_n = d_1 + \dots + d_m$, by the same argument as above, we have the following chain of surjective homomorphisms

$$\begin{aligned} \text{Ind}\left(L(n^{s+t_n}) \boxtimes N\right) &\twoheadrightarrow \text{Ind}\left(L(n^{s+t_n-d_m}) \boxtimes \nabla_{(b_\bullet, b_m)}^{\boxtimes d_m} \boxtimes \nabla_{(n, b_{m-1})}^{\boxtimes d_{m-1}} \boxtimes \dots \boxtimes \nabla_{(n, b_1)}^{\boxtimes d_1}\right) \\ &\simeq \text{Ind}\left(L(n^{s+t_n-d_m}) \boxtimes \nabla_{(n, b_{m-1})}^{\boxtimes d_{m-1}} \boxtimes \dots \boxtimes \nabla_{(n, b_1)}^{\boxtimes d_1} \boxtimes \nabla_{(b_\bullet, b_m)}^{\boxtimes d_m}\right) \\ &\twoheadrightarrow \text{Ind}\left(L(n^{s+t_n-d_m-d_{m-1}}) \boxtimes \nabla_{(n, b_{m-2})}^{\boxtimes d_{m-2}} \boxtimes \dots \boxtimes \nabla_{(n, b_1)}^{\boxtimes d_1} \boxtimes \nabla_{(b_\bullet, b_{m-1})}^{\boxtimes d_{m-1}} \boxtimes \nabla_{(b_\bullet, b_m)}^{\boxtimes d_m}\right) \\ &\vdots \\ &\twoheadrightarrow \text{Ind}\left(L(n^s) \boxtimes \left(\bigboxtimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{\boxtimes d_i}\right)\right). \end{aligned}$$

Therefore, by Lemma 1.8 and [18, Lem. 3.13], we obtain

$$\tilde{f}_n^{s+t_n} \tilde{f}_{n-1}^{t_{n-1}} \dots \tilde{f}_1^{t_1} \mathbf{1} \simeq \text{hd Ind}\left(L(n^{s+t_n}) \boxtimes N\right) \simeq \text{hd Ind}\left(L(n^s) \boxtimes \left(\bigboxtimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{\boxtimes d_i}\right)\right)$$

and this module has multiplicity one as a composition factor of $\text{Ind}\left(L(n^s) \boxtimes \left(\bigboxtimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{\boxtimes d_i}\right)\right)$.

(Case C_n) For type C_n , we write $\mathbf{i}(b_\bullet, b_j) = (n, n-1, \dots, n-l_j+1)$ and let

$$\mathbf{k}_k = (\underbrace{n, \dots, n}_{d_k}, \underbrace{n-1, \dots, n-1}_{d_k}, \dots, \underbrace{n-l_k+1, \dots, n-l_k+1}_{d_k})$$

for $k = 1, \dots, m$. Then \mathbf{k}_k appears in $\text{ch}N_k$ with multiplicity $(d_k!)^{l_k}$. Since $b_\bullet \succ b_m \succ b_{m-1} \succ \dots \succ b_1$, the sequence $\mathbf{k} := \mathbf{k}_1 * \dots * \mathbf{k}_m$ appears in $\text{ch} \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ with multiplicity $(d_1!)^{l_1} \dots (d_m!)^{l_m}$. It follows from Lemma 1.7 and Lemma 4.1 that $\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ is irreducible.

Let $\mathbf{t} = (\underbrace{n, \dots, n}_{t_n}, \underbrace{n-1, \dots, n-1}_{t_{n-1}}, \dots, \underbrace{1, \dots, 1}_{t_1})$. Then we have $e(\mathbf{t})\left(\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)\right) \neq 0$.

Since $\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ is irreducible, by Lemma 1.5 there exists a surjective homomorphism

$$\text{Ind}(L(n^{t_n}) \boxtimes \dots \boxtimes L(1^{t_1})) \twoheadrightarrow \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right).$$

Therefore, it follows from (4.10) that $\tilde{f}_n^{t_n} \dots \tilde{f}_2^{t_2} \tilde{f}_1^{t_1} \mathbf{1} \simeq \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$.

(Case D_n) For $k = 1, \dots, m$, let l_k be the length of $\mathbf{i}(\overline{n-1}, \overline{n}) * \mathbf{i}(\overline{n}, b_k)$ if $\overline{n} \succ b_k$, and $l_k = 1$ if $b_k = n$ or $b_k = \overline{n}$. We will show that $\text{Ind}\left(\bigboxtimes_{k=1}^m \nabla_{(b_\bullet, b_k)}^{\boxtimes d_k}\right)$ is irreducible for type D_n .

First suppose that $b_{m-1} \neq n-1$. Let

$$\mathbf{k}_k = \begin{cases} (\underbrace{n-1, \dots, n-1}_{d_k}) & \text{if } b_k = \overline{n}, \\ (\underbrace{n, \dots, n}_{d_k}, \underbrace{n-1, \dots, n-1}_{d_k}, \dots, \underbrace{n-l_k+1, \dots, n-l_k+1}_{d_k}) & \text{otherwise.} \end{cases}$$

for $k = 1, \dots, m$. Then \mathbf{k}_k appears in $\text{ch}N_k$ with multiplicity $(d_k!)^{l_k}$. Since $b_m \succ b_{m-1} \succ \dots \succ b_1$ and $b_{m-1} \neq n-1$, the sequence $\mathbf{k} := \mathbf{k}_1 * \dots * \mathbf{k}_m$ appears in $\text{ch} \text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ with multiplicity $(d_1!)^{l_1} \dots (d_m!)^{l_m}$. It follows from Lemma 1.7 and Lemma 4.1 that $\text{Ind}\left(\bigboxtimes_{k=1}^m N_k\right)$ is irreducible.

We now assume that $b_{m-1} = n-1$. Without loss of generality, we may suppose that $b_m = \overline{n}$. Since $\text{Ind}(L(n-1) \boxtimes L(n)) \simeq \text{Ind}(L(n) \boxtimes L(n-1))$, we have

$$\text{Ind} \nabla_{(n-1, n-1)}^{\boxtimes d_{m-1}} \boxtimes \nabla_{(n-1, \overline{n})}^{\boxtimes d_m} \simeq \text{Ind} \nabla_{(n-1, \overline{n})}^{\boxtimes d_m} \boxtimes \nabla_{(n-1, n-1)}^{\boxtimes d_{m-1}} \simeq \text{Ind}\left(L(n^{d_{m-1}}) \boxtimes L((n-1)^{d_{m-1}+d_m})\right)$$

and they are irreducible. Let $M_k = \begin{cases} \text{Ind} \nabla_{(b_\bullet, b_k)}^{\boxtimes d_k} & \text{if } k \leq m-2, \\ \text{Ind} \nabla_{(b_\bullet, b_{m-1})}^{\boxtimes d_{m-1}} \boxtimes \nabla_{(b_\bullet, b_m)}^{\boxtimes d_m} & \text{if } k = m-1, \end{cases}$ and set

$$\mathbf{k}_k = \begin{cases} \underbrace{(n, \dots, n, n-1, \dots, n-1, \dots, n-\ell_k+1, \dots, n-\ell_k+1)}_{d_k} & \text{if } k \leq m-2, \\ \underbrace{(n, \dots, n, n-1, \dots, n-1)}_{d_{m-1}} & \text{if } k = m-1. \end{cases}$$

Then the multiplicity of \mathbf{k}_k in $\text{ch}M_k$ is $(d_k!)^{l_k}$ if $k \leq m-2$ and $d_{m-1}!(d_m + d_{m-1})!$ if $k = m-1$. Since $b_\bullet \succ b_m \succ b_{m-1} \succ \dots \succ b_1$, the sequence $\mathbf{k} := \mathbf{k}_1 * \dots * \mathbf{k}_{m-1}$ appears in $\text{ch}(\text{Ind}(M_1 \boxtimes \dots \boxtimes M_{m-1}))$ with multiplicity $(d_1!)^{l_1} \dots (d_{m-2})^{l_{m-2}} (d_{m-1}!)((d_m + d_{m-1})!)$. It follows from Lemma 1.7 and Lemma 4.1 that $\text{Ind}(M_1 \boxtimes \dots \boxtimes M_{m-1})$ is irreducible.

Now for $\mathbf{t} = (\underbrace{n, \dots, n}_{t_n}, \underbrace{n-1, \dots, n-1}_{t_{n-1}}, \dots, \underbrace{1, \dots, 1}_{t_1})$, we have $e(\mathbf{t}) \left(\text{Ind} \left(\bigotimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{d_i} \right) \right) \neq 0$.

Since $\text{Ind} \left(\bigotimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{d_i} \right)$ is irreducible, Lemma 1.5 gives a surjective homomorphism

$$\text{Ind}(L(n^{t_n}) \boxtimes \dots \boxtimes L(1^{t_1})) \twoheadrightarrow \text{Ind} \left(\bigotimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{d_i} \right),$$

which implies the desired conclusion $\tilde{f}_n^{t_n} \dots \tilde{f}_2^{t_2} \tilde{f}_1^{t_1} \mathbf{1} \simeq \text{Ind} \left(\bigotimes_{i=1}^m \nabla_{(b_\bullet, b_i)}^{d_i} \right)$ by (4.10). \square

Recall the definition of $\hat{v} \in \mathbf{B}$ from (3.2). Using the surjective homomorphism in (4.6), we can glue parts (1) and (2) of Lemma 4.2 together for types B_n, C_n , and D_n to get the following result.

Lemma 4.3. *Let $t_i \in \mathbb{Z}_{\geq 0}$ be such that*

$$t_1 \geq t_2 \geq \dots \geq t_{n'-1} \geq t_{n'} \geq t_{n'+1} = 0 \quad (A_n, C_n),$$

$$2t_1 \geq 2t_2 \geq \dots \geq 2t_{n-1} \geq t_n \geq 2t_{n+1} \geq \dots \geq 2t_{n'-1} \geq t_{n'} = 0 \quad (B_n),$$

$$t_1 \geq t_2 \geq \dots \geq t_{n-2} \geq t_{n-1}, t_n \geq t_{n+1} \geq \dots \geq t_{n'-1} \geq t_{n'} = 0 \quad (D_n),$$

where $n' = n$ (A_n), $n' = 2n$ (B_n), $n' = 2n-1$ (C_n, D_n). Set

$$\vartheta_i = \begin{cases} t_i - t_{i+1} & \text{if } i \leq n' \quad (A_n, C_n), \quad i \leq n-2 \quad (B_n), \quad i \leq n-3 \quad (D_n), \\ t_{n-1} - \lceil \frac{t_n}{2} \rceil & \text{if } i = n-1 \quad (B_n), \\ \lceil \frac{t_n}{2} \rceil - \lfloor \frac{t_n}{2} \rfloor & \text{if } i = n \quad (B_n), \\ \lfloor \frac{t_n}{2} \rfloor - t_{n+1} & \text{if } i = n+1 \quad (B_n), \\ t_{n-2} - \max\{t_{n-1}, t_n\} & \text{if } i = n-2 \quad (D_n), \\ \max\{0, t_n - t_{n-1}\} & \text{if } i = n-1 \quad (D_n), \\ \max\{0, t_{n-1} - t_n\} & \text{if } i = n \quad (D_n), \\ \min\{t_{n-1}, t_n\} - t_{n+1} & \text{if } i = n+1 \quad (D_n), \\ t_{i-1} - t_i & \text{if } i \geq n+2 \quad (B_n, D_n). \end{cases}$$

Then

- (1) $\text{hd Ind} \left(\bigotimes_{i=1}^{n'} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right)$ is irreducible and has multiplicity one as a composition factor of $\text{Ind} \left(\bigotimes_{i=1}^{n'} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right)$,
- (2) $\tilde{f}_i \mathbf{1} \simeq \text{hd Ind} \left(\bigotimes_{i=1}^{n'} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right)$, where

$$\mathbf{i} = \begin{cases} (1, \dots, 1, \underbrace{2, \dots, 2}_{t_1}, \dots, \underbrace{n, \dots, n}_{t_n}) & (\mathbf{A}_n), \\ (1, \dots, 1, \dots, \underbrace{n-1, \dots, n-1}_{t_{n-1}}, \underbrace{n, \dots, n}_{t_n}, \underbrace{n-1, \dots, n-1}_{t_{n+1}}, \dots, \underbrace{1, \dots, 1}_{t_{2n-1}}) & (\mathbf{B}_n, \mathbf{C}_n), \\ (1, \dots, 1, \dots, \underbrace{n-2, \dots, n-2}_{t_{n-2}}, \underbrace{n, \dots, n}_{t_{n-1}}, \underbrace{n-1, \dots, n-1}_{t_n}, \dots, \underbrace{1, \dots, 1}_{t_{2n-2}}) & (\mathbf{D}_n). \end{cases}$$

Proof. By Lemma 4.1 and (4.6), for $k \in \mathbb{Z}_{\geq 0}$ we have

$$\begin{aligned} \text{Ind} \left(\nabla_{(\widehat{1,b})}^{\boxtimes k} \boxtimes \nabla_{(b,c)}^{\boxtimes k} \right) &\simeq \text{Ind} \left(\nabla_{(\widehat{1,b})}^{\boxtimes k-1} \boxtimes \nabla_{(\widehat{1,b})} \boxtimes \nabla_{(b,c)} \boxtimes \nabla_{(b,c)}^{\boxtimes k-1} \right) \\ &\rightarrow \text{Ind} \left(\nabla_{(\widehat{1,b})}^{\boxtimes k-1} \boxtimes \nabla_{(\widehat{1,c})} \boxtimes \nabla_{(b,c)}^{\boxtimes k-1} \right) \\ (4.11) \quad &\simeq \text{Ind} \left(\nabla_{(\widehat{1,b})}^{\boxtimes k-1} \boxtimes \nabla_{(b,c)}^{\boxtimes k-1} \boxtimes \nabla_{(\widehat{1,c})} \right) \\ &\vdots \\ &\rightarrow \text{Ind} \left(\nabla_{(\widehat{1,c})}^{\boxtimes k} \right), \end{aligned}$$

where $b \in B(\mathbf{A}_n)$, $b = \widehat{n+1}$ (\mathbf{B}_n), $b = \widehat{n}$ (\mathbf{C}_n), $b = \widehat{n-1}$ (\mathbf{D}_n) and $c \in \mathbf{B}$ with $b \succ c$.

(Case \mathbf{A}_n) The assertion in this case follows from Lemma 4.2.

(Case \mathbf{B}_n) For type \mathbf{B}_n , let

$$\begin{aligned} M(k) &= \text{Ind} \left(\left(\bigotimes_{i=1}^{2n-k} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes (\vartheta_{2n} + \dots + \vartheta_{2n+1-k})} \right), \text{ and} \\ N(k) &= \text{Ind} \left(\nabla_{(\widehat{n+1, 2n+2-k})}^{\boxtimes \vartheta_{2n+1-k}} \boxtimes \nabla_{(\widehat{n+1, 2n+3-k})}^{\boxtimes \vartheta_{2n+2-k}} \boxtimes \dots \boxtimes \nabla_{(\widehat{n+1, 2n+1})}^{\boxtimes \vartheta_{2n}} \right) \end{aligned}$$

for $k = 1, \dots, n$. Note that $t_n = \lceil \frac{t_n}{2} \rceil + \lfloor \frac{t_n}{2} \rfloor$, $\lceil \frac{t_n}{2} \rceil = \vartheta_n + \dots + \vartheta_{2n}$ and $\lfloor \frac{t_n}{2} \rfloor = \vartheta_{n+1} + \dots + \vartheta_{2n}$.

It follows from Lemma 4.2 that

$$M(n) \simeq \tilde{f}_1^{t_1} \dots \tilde{f}_{n-1}^{t_{n-1}} \tilde{f}_n^{\lceil \frac{t_n}{2} \rceil} \mathbf{1}, \quad \text{hd } N(n) \simeq \tilde{f}_n^{\lfloor \frac{t_n}{2} \rfloor} \tilde{f}_{n-1}^{t_{n+1}} \dots \tilde{f}_1^{t_{2n-1}} \mathbf{1}.$$

By Lemma 4.2 and [18, Lem. 3.13], there is a surjective homomorphism

$$\text{Ind} \left(\nabla_{(\widehat{1,n})}^{\boxtimes \lceil \frac{t_n}{2} \rceil} \boxtimes L(n^{\lceil \frac{t_n}{2} \rceil}) \right) \twoheadrightarrow \text{Ind} \nabla_{(\widehat{1,0})}^{\boxtimes \lceil \frac{t_n}{2} \rceil},$$

which yields the following surjective homomorphism

$$\begin{aligned}
& \text{Ind} \left(\left(\left(\bigotimes_{i=1}^{n-1} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,\widehat{n}})}^{\boxtimes \lceil \frac{tn}{2} \rceil} \right) \boxtimes \left(L(n^{\lceil \frac{tn}{2} \rceil}) \boxtimes N(n) \right) \right) \\
& \twoheadrightarrow \text{Ind} \left(\left(\left(\bigotimes_{i=1}^{n-1} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes \lceil \frac{tn}{2} \rceil} \right) \boxtimes N(n) \right) \\
& \simeq \text{Ind} \left(\left(\left(\bigotimes_{i=1}^n \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes \lfloor \frac{tn}{2} \rfloor} \right) \boxtimes N(n) \right) \\
& \simeq \text{Ind}(M(n) \boxtimes N(n)).
\end{aligned}$$

It follows from Lemma 4.2 that

$$\begin{aligned}
\tilde{f}_1^{t_1} \cdots \tilde{f}_{n-1}^{t_{n-1}} \mathbf{1} & \simeq \text{Ind} \left(\left(\left(\bigotimes_{i=1}^{n-1} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,\widehat{n}})}^{\boxtimes \lceil \frac{tn}{2} \rceil} \right) \right), \\
\tilde{f}_n^{t_n} \tilde{f}_{n-1}^{t_{n+1}} \cdots \tilde{f}_1^{t_{2n-1}} \mathbf{1} & \simeq \text{hd Ind} \left(L(n^{\lceil \frac{tn}{2} \rceil}) \boxtimes N(n) \right).
\end{aligned}$$

Since $\varepsilon_i \left(\text{Ind}(L(n^{\lceil \frac{tn}{2} \rceil}) \boxtimes N(n)) \right) = 0$ for $i = 1, \dots, n-1$, by Lemma 1.8 we have

$$\tilde{f}_i \mathbf{1} \simeq \text{hd Ind}(M(n) \boxtimes N(n)),$$

and this module has multiplicity one as a composition factor of $\text{Ind}(M(n) \boxtimes N(n))$.

Now by Lemma 4.1 and (4.11), we have the following chain of surjective homomorphisms

$$\begin{aligned}
& \text{Ind}(M(n) \boxtimes N(n)) \\
& \simeq \text{Ind} \left(\left(\left(\bigotimes_{i=1}^n \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes (\vartheta_{2n} + \cdots + \vartheta_{n+2})} \right) \boxtimes \left(\nabla_{(\widehat{1,n+1})}^{\boxtimes \vartheta_{n+1}} \boxtimes \nabla_{(\widehat{n+1,n+2})}^{\boxtimes \vartheta_{n+1}} \right) \boxtimes N(n-1) \right) \\
& \twoheadrightarrow \text{Ind} \left(\left(\left(\bigotimes_{i=1}^n \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes (\vartheta_{2n} + \cdots + \vartheta_{n+2})} \right) \boxtimes \nabla_{(\widehat{1,n+2})}^{\boxtimes \vartheta_{n+1}} \right) \boxtimes N(n-1) \\
& \simeq \text{Ind}(M(n-1) \boxtimes N(n-1)) \\
& \quad \vdots \\
& \twoheadrightarrow \text{Ind}(M(1) \boxtimes N(1)) \simeq \text{Ind} \left(\left(\left(\bigotimes_{i=1}^{2n-1} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1,n+1})}^{\boxtimes \vartheta_{2n}} \right) \boxtimes \nabla_{(\widehat{n+1,2n+1})}^{\boxtimes \vartheta_{2n}} \right) \\
& \twoheadrightarrow \text{Ind} \left(\left(\bigotimes_{i=1}^{2n} \nabla_{(\widehat{1,i+1})}^{\boxtimes \vartheta_i} \right) \right),
\end{aligned}$$

which completes the proof for B_n .

(Case C_n) For type C_n , set

$$M(k) = \text{Ind} \left(\left(\bigotimes_{i=1}^{n-1} \nabla_{(\widehat{1, i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1, \widehat{n}})}^{\boxtimes (\vartheta_n + \dots + \vartheta_{n-1+k})} \right), \quad \text{and}$$

$$N(k) = \text{Ind} \left(\nabla_{(\widehat{\widehat{n}, k+n})}^{\boxtimes \vartheta_{k+n-1}} \boxtimes \nabla_{(\widehat{\widehat{n}, k+n-1})}^{\boxtimes \vartheta_{k+n-2}} \boxtimes \dots \boxtimes \nabla_{(\widehat{\widehat{n}, n+1})}^{\boxtimes \vartheta_n} \right)$$

for $1 \leq k \leq n$. It follows from Lemma 4.2 that

$$M(n) \simeq \tilde{f}_1^{t_1} \tilde{f}_2^{t_2} \dots \tilde{f}_{n-1}^{t_{n-1}} \mathbf{1}, \quad N(n) \simeq \tilde{f}_n^{t_n} \tilde{f}_{n-1}^{t_{n+1}} \dots \tilde{f}_1^{t_{2n-1}} \mathbf{1}.$$

Since $\varepsilon_i(N(n)) = 0$ for $i = 1, \dots, n-1$, Lemma 1.8 implies

$$\tilde{f}_1 \mathbf{1} \simeq \text{hd Ind}(M(n) \boxtimes N(n))$$

and this module has multiplicity one as a composition factor in $\text{Ind}(M(n) \boxtimes N(n))$.

By Lemma 4.1 and (4.11), we have a chain of surjective homomorphisms

$$\begin{aligned} \text{Ind}(M(n) \boxtimes N(n)) &\simeq \text{Ind} \left(M(n-1) \boxtimes \nabla_{(\widehat{1, \widehat{n}})}^{\boxtimes \vartheta_{2n-1}} \boxtimes \nabla_{(\widehat{\widehat{n}, 2n})}^{\boxtimes \vartheta_{2n-1}} \boxtimes N(n-1) \right) \\ &\rightarrow \text{Ind} \left(M(n-1) \boxtimes \nabla_{(\widehat{1, 2n})}^{\boxtimes \vartheta_{2n-1}} \boxtimes N(n-1) \right) \\ &\simeq \text{Ind} \left(M(n-1) \boxtimes N(n-1) \boxtimes \nabla_{(\widehat{1, 2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\quad \vdots \\ &\rightarrow \text{Ind} \left(M(1) \boxtimes N(1) \boxtimes \nabla_{(\widehat{1, n+2})}^{\boxtimes \vartheta_{n+1}} \boxtimes \dots \boxtimes \nabla_{(\widehat{1, 2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\rightarrow \text{Ind} \left(\left(\bigotimes_{i=1}^{n-1} \nabla_{(\widehat{1, i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1, n+1})}^{\boxtimes \vartheta_n} \boxtimes \dots \boxtimes \nabla_{(\widehat{1, 2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\simeq \text{Ind} \left(\bigotimes_{i=1}^{2n-1} \nabla_{(\widehat{1, i+1})}^{\boxtimes \vartheta_i} \right), \end{aligned}$$

which yields $\tilde{f}_1 \mathbf{1} \simeq \text{hd Ind} \left(\bigotimes_{i=1}^{2n-1} \nabla_{(\widehat{1, i+1})}^{\boxtimes \vartheta_i} \right)$.

(Case D_n) Without loss of generality, we may assume in the D_n case that $t_{n-1} \geq t_n$. Note that $\vartheta_{n-1} = 0$ and $\vartheta_n = \max\{t_{n-1}, t_n\} - \min\{t_{n-1}, t_n\}$. Let

$$M(k) = \text{Ind} \left(\left(\bigotimes_{i=1}^{n-2} \nabla_{(\widehat{1, i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1, n-1})}^{\boxtimes (\vartheta_n + \dots + \vartheta_{n-1+k})} \right),$$

$$N(k) = \text{Ind} \left(\nabla_{(\widehat{\widehat{n-1}, k+n})}^{\boxtimes \vartheta_{k+n-1}} \boxtimes \nabla_{(\widehat{\widehat{n-1}, k+n-1})}^{\boxtimes \vartheta_{k+n-2}} \boxtimes \dots \boxtimes \nabla_{(\widehat{\widehat{n-1}, n+1})}^{\boxtimes \vartheta_n} \right)$$

for $k = 1, \dots, n$. It follows from Lemma 4.2 that

$$M(n) \simeq \tilde{f}_1^{t_1} \tilde{f}_2^{t_2} \dots \tilde{f}_{n-2}^{t_{n-2}} \mathbf{1}, \quad N(n) \simeq \tilde{f}_n^{t_{n-1}} \tilde{f}_{n-1}^{t_n} \tilde{f}_{n-2}^{t_{n+1}} \dots \tilde{f}_1^{t_{2n-2}} \mathbf{1}.$$

Since $\varepsilon_i(N(n)) = 0$ for $i = 1, \dots, n-2$, Lemma 1.8 gives

$$\tilde{f}_i \mathbf{1} \simeq \text{hd Ind}(M(n) \boxtimes N(n)),$$

and this module with multiplicity one in $\text{Ind}(M(n) \boxtimes N(n))$.

Since $\nabla_{(\widehat{1}, \widehat{n})}^{\boxtimes \vartheta_{n-1}} = \mathbb{C}$, Lemma 4.1 and (4.11) again give a string of surjective homomorphisms

$$\begin{aligned} \text{Ind}(M(n) \boxtimes N(n)) &\simeq \text{Ind} \left(M(n-1) \boxtimes \nabla_{(\widehat{1}, \widehat{n-1})}^{\boxtimes \vartheta_{2n-1}} \boxtimes \nabla_{(\widehat{n-1}, \widehat{2n})}^{\boxtimes \vartheta_{2n-1}} \boxtimes N(n-1) \right) \\ &\twoheadrightarrow \text{Ind} \left(M(n-1) \boxtimes \nabla_{(\widehat{1}, \widehat{2n})}^{\boxtimes \vartheta_{2n-1}} \boxtimes N(n-1) \right) \\ &\simeq \text{Ind} \left(M(n-1) \boxtimes N(n-1) \boxtimes \nabla_{(\widehat{1}, \widehat{2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\quad \vdots \\ &\twoheadrightarrow \text{Ind} \left(M(1) \boxtimes N(1) \boxtimes \nabla_{(\widehat{1}, \widehat{n+2})}^{\boxtimes \vartheta_{n+1}} \boxtimes \dots \boxtimes \nabla_{(\widehat{1}, \widehat{2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\twoheadrightarrow \text{Ind} \left(\left(\boxtimes_{i=1}^{n-2} \nabla_{(\widehat{1}, \widehat{i+1})}^{\boxtimes \vartheta_i} \right) \boxtimes \nabla_{(\widehat{1}, \widehat{n+1})}^{\boxtimes \vartheta_n} \boxtimes \dots \boxtimes \nabla_{(\widehat{1}, \widehat{2n})}^{\boxtimes \vartheta_{2n-1}} \right) \\ &\simeq \text{Ind} \left(\boxtimes_{i=1}^{2n-1} \nabla_{(\widehat{1}, \widehat{i+1})}^{\boxtimes \vartheta_i} \right), \end{aligned}$$

which yields the desired result $\tilde{f}_i \mathbf{1} \simeq \text{hd Ind} \left(\boxtimes_{i=1}^{2n-1} \nabla_{(\widehat{1}, \widehat{i+1})}^{\boxtimes \vartheta_i} \right)$. \square

As an immediate consequence of this lemma we have

Corollary 4.4. $\mathcal{N}_n(v) = \text{hd Ind} \nabla(\mathbf{a}(v); n)$ for $v \in B(\infty)$, where $\mathcal{N}_n(v)$ is the irreducible module given in Proposition 2.3.

We now are ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $I_{(k)} = \{n-k+1, n-k+2, \dots, n\}$ for $k = 1, \dots, n$. Note that $I_{(k)} \subset I_{(k+1)}$ and $|I_{(k)}| = k$. Let $U_q(\mathfrak{g}_k)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by e_i, f_i ($i \in I_{(k)}$) and q^h ($h \in \mathbb{P}^\vee$), and let \mathfrak{B}_k be the crystal obtained from $B(\infty)$ by forgetting the i -arrows for $i \notin I_{(k)}$. It follows from Table 1 that $U_q(\mathfrak{g}_k)$ is of type \mathbf{X}_k when $U_q(\mathfrak{g})$ is of type \mathbf{X}_n ($\mathbf{X} = \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$). Recall the definition \mathbf{a}_i for $\mathbf{a} \in \mathcal{S}$ and the sequences \mathbf{s}_k given in Section 2. Take $v \in B(\infty)$ and let $\triangle(\mathbf{a}(v)) = \{t_{ij}\}$. If $i = 1, \dots, n$ for type $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n$, then

$$\mathbf{a}(v)_i = (t_{i, n+1-i}, t_{i, n+2-i}, \dots, t_{i, d_i}),$$

where $d_i = n$ for all i (\mathbf{A}_n) and $d_i = n-1+i$ ($\mathbf{B}_n, \mathbf{C}_n$). If $i = 1, \dots, n$ for type \mathbf{D}_n , then

$$\mathbf{a}(v)_i = \begin{cases} (t_{1, n-1}) & \text{if } i = 1, \\ (t_{1, n}) & \text{if } i = 2, \\ (t_{i-1, n+1-i}, t_{i-1, n+2-i}, \dots, t_{i-1, n-2+i}) & \text{if } i = 3, \dots, n. \end{cases}$$

Let $M_k = \text{Ind} \nabla(\mathbf{a}(v); k)$ for $k = 1, \dots, n$. Then, by Proposition 2.3, Lemma 4.3, and the choice of $I_{(k)}$, we obtain

$$\mathcal{N}_k(v) = \text{hd} M_k \quad \text{for } k = 1, \dots, n.$$

By the construction of $\nabla(\mathbf{a}(v); k)$ and Lemma 4.3, we know

- (i) $\varepsilon_i(M_k) = 0$ for $i \in I_{(k-1)}$,
- (ii) $\text{hd} M_k$ is irreducible and occurs with multiplicity one as a composition factor of M_k .

Therefore, by Lemma 1.8 and Proposition 1.10,

$$\begin{aligned} \Phi(v) &= \text{hd Ind}((\text{hd} M_1) \boxtimes (\text{hd} M_2) \boxtimes \cdots \boxtimes (\text{hd} M_n)) \\ &= \text{hd Ind}(M_1 \boxtimes M_2 \boxtimes \cdots \boxtimes M_n) \\ &= \text{hd Ind}(\nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)), \end{aligned}$$

which completes the proofs of (1) and (2).

Let $\Psi^\lambda : B(\lambda) \rightarrow \mathbb{B}(\lambda)$ be the canonical crystal isomorphism given by $\Psi^\lambda(\tilde{f}_{\mathbf{i}} b_\lambda) = \tilde{f}_{\mathbf{i}} \mathbf{1}$ for any sequence \mathbf{i} of elements in I . Then the following diagram commutes:

$$\begin{array}{ccc} B(\lambda) & \xrightarrow[\Psi^\lambda]{\sim} & \mathbb{B}(\lambda) \\ \downarrow \iota_\lambda & & \downarrow j_\lambda \\ B(\infty) \otimes T_\lambda \otimes C & \xrightarrow[\Psi \otimes \text{id} \otimes \text{id}]{\sim} & \mathbb{B}(\infty) \otimes T_\lambda \otimes C \end{array}$$

Here, ι_λ (resp. j_λ) is the crystal embedding (1.1) from $B(\lambda)$ (resp. $\mathbb{B}(\lambda)$) to $B(\infty) \otimes T_\lambda \otimes C$ (resp. $\mathbb{B}(\infty) \otimes T_\lambda \otimes C$). Assume for $v \in B(\lambda)$ that $\iota_\lambda(v) = v' \otimes t_\lambda \otimes c$ for some $v' \in B(\infty)$, and recall that $\mathbf{a}(v) = \mathbf{a}(v')$. Let j_λ^{-1} be the inverse from $\text{im}(j_\lambda)$ to $\mathbb{B}(\lambda)$. Then

$$\Psi^\lambda = j_\lambda^{-1} \circ (\Psi \otimes \text{id} \otimes \text{id}) \circ \iota_\lambda,$$

which yields

$$\begin{aligned} \Psi^\lambda(v) &= j_\lambda^{-1} \circ (\Psi \otimes \text{id} \otimes \text{id}) \circ \iota_\lambda(v) \\ &= j_\lambda^{-1} \circ (\Psi \otimes \text{id} \otimes \text{id})(v' \otimes t_\lambda \otimes c) \\ &= j_\lambda^{-1}(\text{hd Ind}(\nabla(\mathbf{a}(v'); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v'); n)) \otimes t_\lambda \otimes c) \\ &= j_\lambda^{-1}(\text{hd Ind}(\nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)) \otimes t_\lambda \otimes c) \\ &= \text{hd Ind}(\nabla(\mathbf{a}(v); 1) \boxtimes \cdots \boxtimes \nabla(\mathbf{a}(v); n)). \end{aligned}$$

This proves assertion (3) and concludes the proof of the main theorem (Theorem 3.2). \square

Acknowledgments

Work on this paper was facilitated by a visit by S.-J. Kang, S.-j. Oh, and E. Park to the University of Wisconsin-Madison and by a visit by G. Benkart to the Korea Institute for Advanced Study. We express our gratitude to these institutions for their hospitality.

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